Mathematical reasoning competency and digital tools

A design research study in lower secondary school on using sliders for variable expressions relating GeoGebra's graphic and algebra view within justification processes

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The dissertation is article based, consisting of the following six research papers and a summarizing report, framing the project.

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Paper 2 Gregersen, R. M., & Baccaglini-Frank, A. (2020). Developing an analytical tool of the processes of justificational mediation. In A. Donevska-Todorova, E. Faggiano, J. Trgalova, Z. Lavicza, R. Weinhandl, A. Clark-Wilson, & H.-G. Weigand (Eds.), *Proceedings of the Tenth ERME Topic Conference (ETC 10) on Mathematics Education in the Digital Age (MEDA), 16-18 September 2020 in Linz, Austria* (pp. 451–458). https://hal.archives-ouvertes.fr/hal-02932218

Paper 3 Gregersen, R. M., & Baccaglini-Frank, A. (2022). Lower secondary students' reasoning competency in a digital environment: the case of instrumented justification. In U. T. Jankvist & E. Geraniou (Eds.), *Mathematical Competencies in the Digital Era* (pp. 119–138). Springer, Cham. https://doi.org/10.1007/978-3-031-10141-0_2

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Paper 6 Bach, C. C., Pedersen, M. K., Gregersen, R. M., & Jankvist, U. T. (2021). On the notion of ”background and foreground” in networking of theories. In Y. Liljekvist, L. B. Boistrup, J. Häggström, L. Mattsson, O. Olande, & H. Palmér (Eds.), *Sustainable mathematics education in a digitalized world: Proceedings of The twelfth research seminar of the Swedish Society for Research in Mathematics Education (MADIF 12) Växjö, January 14–15, 2020* (Vol. 15, pp. 163–172). Skrifter från Svensk Förening för Matematik Didaktisk Forskning. http://matematikdidaktik.org/index.php/madifs-skriftserie/

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SUMMARY

In recent decades, among others, two areas of focus have emerged in mathematics education research. The first emphasizes mastering mathematics as competencies, proficiency, or literacy, as opposed to merely knowing mathematical facts and skills (Niss, 2016). The second area explores the use of digital technologies in teaching and learning mathematics (Artigue & Trouche, 2021).

The Danish competency framework, known as the KOM-framework, outlines eight mathematical competencies that describe activities related to doing and dealing with mathematics (Niss & Højgaard, 2019). Among the eight competencies, mathematical reasoning competency, which involves analyzing or producing arguments to justify mathematical claims, is the focus of this project (Niss & Højgaard, 2019). The KOM-framework, along with the use of digital tools, is featured in the Danish mathematics curricula across primary, lower secondary, and upper secondary education (Danmarks Evalueringsinstitut [EVA] 2009). In addition, digital mathematics tools, such as dynamic geometry environments and computer algebra systems, increasingly integrate functionalities from both geometry and algebra (Freiman, 2014; Sutherland & Rojano, 2014), offering new possibilities and complexities that often surpass the understanding of laypersons. This study reported in this dissertation has both practical and theoretical aims. Practically, it investigates how the integration of digital tools in geometry and algebra can support students' reasoning processes in lower secondary mathematics education and emphasizes enabling students to exercise their mathematical reasoning competency rather than developing the competency as such. Theoretically, the study seeks to promote sustainable theoretical development by linking the KOM-framework with international mathematics education research. This is achieved by adopting a networking perspective (Prediger, Bikner-Ahsbahs, et al., 2008) on theory development.

Design research (Bakker, 2018; Cobb et al., 2003; Gravemeijer & Prediger, 2019; McKenney & Reeves, 2014). as the methodological framework for this project, guiding the collection and analysis of data, as well as the establishment of learning situations that incorporate mathematical reasoning competency, digital technologies, and the variable as a generalized number. A microworlds (Hoyles, 1993) of variable points along with task sequences were developed, aiming for students to exercise mathematical reasoning competency by investigating basic algebraic expressions and their structural implications in the dynamic behavior of variable points.

In addition to the KOM-framework, the project employs other theoretical perspectives, such as the instrumental approach to mathematics education (IAME) (Guin & Trouche, 1998) and the elaborated notion of a scheme (Vergnaud, 1998b) in relation to the scheme-technique duality (Drijvers et al., 2013), and Toulmin's (2003) argumentation model. These perspectives are used for analyzing,

describing, and explaining empirical data, particularly regarding students' use of digital tools and their mathematical reasoning competency.

The dissertation comprises six research papers and an accompanying kappa that provides the theoretical background, methodology, additional analysis, and results. Paper 1 is a literature review that identifies potential tools within GeoGebra, a dynamic geometry and algebra environment, for student justifications of the variable as a generalized number, which informed the task design. Papers 2 through 5 analyze empirical data from students working with different tasks and the development of an analytical tool for instrumented justification. Paper 2 introduces the initial connection between the KOM and IAME frameworks, interpreted through Toulmin's model, leading to the creation of an analytical tool. Paper 3 refines this tool, defines instrumented justification, and describes students' justification processes using artifacts. Paper 4 focuses on the goal of student tool use within a task that is further developed, examining its potential and challenges for exercising reasoning competency. Paper 5 emphasizes the scheme aspect of the analytical tool, analyzing Vergnaud's (1998) scheme components and elaborating on how students' conceptual knowledge integrates into their instrumented justification processes. Paper 6 addresses the notion of foreground and background theory within the networking of theories perspective.

The dissertation contributes three design principles for task design that support students' exercise of reasoning competency, a microworld for exploring and justifying the dynamic behavior of variable points, and associated tasks. It also elaborates on reasoning competency in students' instrumented justification processes and the scheme-technique duality and provides suggestions for supporting these processes in the classroom and through task design. Additionally, it identifies a hybrid conception between continuous and discrete understandings of variables in predicting variable behavior within the dynamic geometry and algebra environment and suggests theoretical links between the KOM and IAME frameworks as potentials for further theoretical networking.

RESUMÉ

I de seneste årtier er to fokusområder løbende blevet diskuteret inden for forskningen i matematikdidaktik. Det første fokusområde italesætter at mestre og lære matematik som besiddelse og udvikling af matematisk kompetence i stedet for viden og læring af matematiske fakta og færdigheder (Niss, 2016). Det andet område udforsker brugen af digitale teknologier i undervisning og læring af matematik (Artigue & Trouche, 2021). Den danske KOM-rapport skitserer otte matematiske kompetencer, der beskriver aktiviteter relateret til at udøve og håndtere matematik (Niss & Højgaard, 2019). Blandt de otte kompetencer er matematisk ræsonnementskompetence, som er omdrejningspunkt for dette studie. Ræsonnementskompetence indebærer at analysere eller producere argumenter for at begrunde matematiske påstande (Niss & Højgaard, 2019). Kompetencerne, sammen med brugen af digitale værktøjer, er en del af de danske matematiklæreplaner på både grundskole- og gymnasieniveau (Danmarks Evalueringsinstitut [EVA], 2009).

Moderne digitale matematikværktøjer, såsom dynamiske geometri programmer og computer algebra systemer (CAS), integrerer i stigende grad funktionalitet fra hinanden (Freiman, 2014; Sutherland & Rojano, 2014), hvilket giver nye muligheder men også øger programmernes kompleksitet. Dette studie, rapporteret i denne afhandling, har både praktiske og teoretiske formål. Praktisk undersøges, hvordan integrationen af geometri og algebra i digitale matematikværktøjer kan støtte elevers ræsonnementsprocesser i udskolingens matematikundervisning. Studiet lægger vægt på at gøre det muligt for eleverne at udøve deres matematiske ræsonnementskompetence snarere end som sådan at udvikle elevernes kompetence. Teoretisk søger studiet at fremme bæredygtig teoretisk udvikling ved at forbinde KOM med international matematikdidaktiskforskning. Dette opnås ved at anvende et netværksperspektiv (Prediger, Bikner-Ahsbahs et al., 2008) på teoriudvikling.

Projektets metodiske udgangspunkt er designbaseret forskning (Bakker, 2018; Cobb et al., 2003; Gravemeijer & Prediger, 2019; McKenney & Reeves, 2014). Det har guidet indsamling og analyse af data samt design af opgaver, der fordrer elevers udøvelse af matematiske ræsonnementskompetence i brugen af digitale teknologier med fokus på variable som et generaliseret tal. I den henseende er der udviklet og designet en "microworld" (Hoyles, 1993) med variable punkter med tilhørende opgavesekvenser. Den er udviklet med henblik på at lade eleverne udøve matematiske ræsonnementskompetencer gennem deres undersøgelse af grundlæggende algebraiske udtryk og strukturelle implikationer i de variable punkters dynamiske egenskaber.

Udover KOM anvender projektet andre teoretiske perspektiver, såsom den instrumentelle tilgang til matematikundervisning (IAME) (Guin & Trouche, 1998) og dens opfattelse af kognitive skemaer (Vergnaud, 1998b) i relation til skema-teknik-dualiteten (Drijvers et al., 2013), samt Toulmins (2003)

argumentationsmodel. Disse perspektiver bruges til at analysere, beskrive og forklare empiriske data, særligt i forhold til elevernes brug af digitale værktøjer og deres matematiske ræsonnementskompetence.

Afhandlingen består af seks forskningsartikler og denne tilhørende rapport, der beskriver den teoretiske baggrund, metodologi samt bidrager med yderligere analyser og resultater. Paper 1 er et litteraturstudie, der identificerer potentielle værktøjer i GeoGebra, et dynamisk geometri- og algebraprogram, til elevers ræsonnementer af variablen som et generaliseret tal. Paper 1 har informeret studiets efterfølgende designprocesser og produkter. Papers 2 til 5 analyserer empiriske data fra elever, der arbejder med forskellige opgaver, og udviklingen af et analytisk værktøj til instrumented justification. Artikel 2 introducerer den indledende teoretiske udvikling i at forbinde KOM og IAME, som genfortolkes gennem Toulmins model, hvilket fører til udviklingen af et analytisk værktøj. Artikel 3 forfiner dette værktøj, instrumented justification, og beskriver elevers ræsonnementsprocesser ved brug af digitale værktøjer. Artikel 4 fokuserer på elevers mål i brugen af digitale værktøjer i deres undersøgelser og løsning af en opgave. Opgaven videreudvikles på baggrund heraf, og opgavens potentiale og udfordringer for udøvelse af ræsonnementskompetence undersøges. Artikel 5 fokuserer på kognitive skemaer i det analytiske værktøj i analyser af Vergnauds (1998) skema-bestanddele og uddyber, hvordan elevers konceptuelle viden integreres i deres instrumenterede ræsonnementsprocesser. Artikel 6 behandler begrebet forgrunds- og baggrundsteori i et netværksperspektiv på teoriudvikling.

Afhandlingen bidrager med tre designprincipper for opgavedesign, der understøtter elevers udøvelse af ræsonnementskompetence, en ”microworld” til at udforske og begrunde variable punkters dynamiske bevægelse og tilhørende opgaver. Den uddyber også ræsonnementskompetence i elevers instrumenterede ræsonnementsprocesser og skema-teknik-dualiteten og giver forslag til at understøtte disse processer i klasseværelset gennem opgavedesigns. Derudover identificerer den en hybridopfattelse mellem kontinuerlige og diskrete forståelser af variable i forudsigelsen af variable punkters bevægelsesmønstre i dynamiske geometri- og algebraprogrammer og foreslår teoretiske forbindelser mellem KOM og IAME som potentialer for yderligere teoretisk udvikling.

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1 INTRODUCTION

Digital mathematics tools have become an integrated and widespread part of mathematics education, originating a new paradigm for mathematics education (Mullis et al., 2016; Trouche et al., 2013). Dynamic geometry environments (DGE) are predominant in mathematics education in primary and lower secondary education (Mullis et al., 2016), whereas computer algebra systems (CAS) are the main tool in upper secondary education in Denmark (Grønbæk et al., 2017; Trouche, 2005). However, the digital mathematics tools of today increasingly draw on functionalities from one another, relating the two domains of geometry and algebra. This development renders both new possibilities and increased complexity beyond the comprehension of laymen.

Alongside the increase of digital tools, the KOM framework (in short, KOM) was introduced in Denmark with the *Competencies and Mathematical Learning* report from 2002 (Niss & Højgaard, 2011; Niss & Jensen, 2002). It compelled a shift in mathematics education in Denmark from understanding mathematics and mathematics education to be a matter of skill and knowledge to one of mathematical competence and mathematical competencies[1] (Niss & Jankvist, 2022). Among the eight competencies, we have only just started to understand how digital tools influence and interplay with students' development of competencies.

The present PhD project studies lower secondary students' mathematical reasoning competency (RC) (Niss & Højgaard, 2019) in situations involving mathematical digital technologies that merge CAS functionality into a DGE – resulting in a dynamic geometry and algebra environment (DGAE). The impact of digital technologies on mathematical development and reasoning has been a significant topic in mathematics education research (Artigue, 2010; Trouche et al., 2013). Niss (2016) argues that digital tools can either enhance or replace mathematical competencies, depending on their use. This emphasizes the educational value of tool use for educational prospects residing within explorational and interpretational use of tools (Artigue, 2002), which is fundamental to reasoning processes (Misfeldt & Jankvist, 2019). The practical aim of this study is to address how DGAEs can play such a role in students' RC in lower secondary mathematics education.

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[1] KOM differentiates between competenc*e* and competenc*y*. While mathematical competenc*e* involves using mathematics to tackle various challenges, mathematical competenc*y* focuses on addressing specific types of challenges requiring particular mathematical skills. Thus, mathematical competence is built from a set of mathematical competencies.

The KOM framework has been developed within a national context, so in order to explore both RC and digital tools, it is necessary to relate the framework to international mathematics education research (MER). MER encompasses a diversity of concepts, frameworks, and theories that have originated and evolved within a diverse research community (Prediger, Bikner-Ahsbahs, et al., 2008) and may not necessarily incorporate a competency perspective. Therefore, extending the boundaries of a framework developed at a national level to incorporate theories from the internal research community requires careful consideration to maintain the original identity of the KOM framework.

The research practice of networking theories (Prediger, Bikner-Ahsbahs, et al., 2008) offers both perspectives on reflective practices for connecting theoretical approaches as well as strategies to integrate theoretical approaches. Linking RC with other theoretical approaches resembles what Niss and Jankvist (2022) describe as mutual fertilization and can suggest potentials of such integration. With this in mind, the overarching goal of the study is to promote sustainable theoretical development that links the KOM framework with theories in MER, by addressing the practical aim of investigating the potential of DGAEs for students to exercise their RC.

1.1 MATHEMATICAL COMPETENCIES IN MER AND CURRICULA

Already in 1973, McClelland advocated for assessing education with regard to competence instead of intelligence. Since then, the concept of competence has gained significant traction, particularly in curricula and educational research. Numerous competency frameworks have emerged, such as the 21st century skills (Berthelsen, 2017; Partnership for 21st Century Skills 2002), The general term *competency* can be akin to "ability, capability, cognizance, effectuality, efficacy, efficiency, knowledge, mastery, proficiency, skill, and talent" (Kilpatrick, 2014, p. 85). The Danish response to the competency paradigm originated with the KOM project, resulting in the KOM report in 2002 (Niss & Jensen, 2002) (Jensen is now Højgaard). This report has been translated to English by Niss and Højgaard (2011) and revisited in Niss and Højgaard (2019).

In all, the KOM framework describes eight distinct, yet mutually related, mathematical competencies; the one of relevance in this study is the RC (Niss & Højgaard, 2011, 2019). The framework has had significant influence on the education system in Denmark but has also "generated extensive discussion and a multitude of additional conceptual developments, oftentimes in connection with different theoretical, empirical and practical uses of the notions, as reflected in many publications" (Niss & Højgaard, 2019, p. 2). Højsted (2021), Thomsen (2022), Bach (2022) and (Pedersen, 2024) are all examples of this, and international collaborations have resulted in the book *Mathematical Competencies in the Digital Era* (Jankvist & Geraniou, 2022). Moreover, the research project "The Didactics of 21st Century Mathematics Teaching and Learning or

Mathematical Competencies and ICT", in which this PhD project is rooted, aims to connect theories from MER to the KOM framework to better understand the interplay of digital tools and students' mathematical competencies.

1.1.1 Reasoning competency

RC has undergone significant development from the original framework (Niss & Højgaard, 2011; Niss & Jensen, 2002) to Niss and Højgaard (2019), as the emphasis has changed from proof to justification. This is both evident in the definition and description. Originally, the authors wrote that RC is:

> on the one hand, the ability to *follow* and *assess mathematic reasoning*, i.e. a chain of argument put forward by others, in writing or orally, in support of a claim. It is especially about knowing and *understanding* what a mathematical *proof* is and how this differs from other forms of mathematical reasoning. (Niss & Højgaard, 2011, p. 60)

The revised version of RC, however, is defined as "to analyze or produce arguments (i.e., chains of statements linked by inferences) put forward in oral or written form to justify mathematical claims" (Niss & Højgaard, 2019, p. 16). Additionally, the early description mentioned *proof* on four separate occasions, and only hints to other reasoning forms as "informal arguments". In the revised description, *proof* appears once and *justification* four times, and it is stressed that "the kinds of claims at issue in this competency are not confined to "theorems" or "formulae" but comprise all sorts of conclusions obtained by mathematical methods and inferences, including solutions to problems" (Niss & Højgaard, 2019, p. 16).

Reasoning as proof only receives little attention in the lower secondary education (Ministry of Children and Education, 2019), whereas justification is applicable to other and more occurrent processes in mathematics teaching and learning at this educational stage, such as problem-solving processes. The prior emphasis on proof in RC is also reflected in the research on RC at lower secondary education and the use of technology, for example Højsted (2021) and Thomsen (2022), who both regard aspects of proof. The shift in emphasis with respect to RCs has implication for lower secondary education. Otherwise, this study emphasizes justification processes, aligning with the revised description of RC, and hence contributes with new aspects to our understating of students' use of tools in relation to the competency.

1.2 THE USE OF TOOLS IN EDUCATION AND MER

MER on digital tools in mathematics education has developed from the ideas and ideals of Papert (1980), which materialized in the turtle programming software LOGO, to the study of DGE, CAS

(Villa-Ochoa & Suárez-Téllez, 2021) and microworlds (Edwards, 1998). Throughout the last decade, DGE and CAS software has adopted features from each other, providing symbolic manipulation of geometric construction and graphs, plots, etc. (Hohenwarter & Jones, 2007). The one mathematics software that has taken the development to the full extent is GeoGebra, which

> provides a closer connection between the symbolic manipulation and visualisation capabilities of CAS and the dynamic changeability of DGE. It does this by providing not only the functionality of DGE (in which the user can work with points, vectors, segments, lines, and conic sections) but also of CAS (in that equations and coordinates can be entered directly and functions can be defined algebraically and then changed dynamically. (Hohenwarter & Jones, 2007, p. 127)

As such, GeoGebra is a digital environment that combines the traditional features of a DGE with the algebraic features of a CAS tool. This is, for example, evident in the so-called algebra view, where symbolic representations of items in the graphic view can be constructed and manipulated. Furthermore, GeoGebra has been developed as an educational tool (which is common for DGEs) rather than an expert tool (which is often the case for CAS). In the Danish educational system GeoGebra is widely implemented in both primary and lower secondary school mathematics education (Højsted, 2020b). GeoGebra is widespread as it is free, open source (Sutherland & Rojano, 2014), and has been translated into many languages.

Affordances particularly associated with the integration of geometry and algebra are the dynamic characters of multi-representation, which can "contribute to improvement in the reasoning, understanding, and conceptualization of mathematical objects" (Villa-Ochoa & Suárez-Téllez, 2021, p. 5). Nevertheless, the potential of the algebra view for the learning and teaching of mathematics, and in particular students' development of RC, is yet to be fully unveiled (e.g., Hohenwarter & Jones, 2007).

The use of DGE and CAS allows for the exploration and investigation of mathematical concepts through access to multi-representation (Drijvers et al., 2009). However, this potential is not automatically realized, especially when students encounter challenges in using the tools or understanding the mathematical content, deflecting the students' focus from mathematical reflection (Guin & Trouche, 1998). In MER, there are two dominating theories concerning students' use of tools: the Instrumental Approach to Mathematics Education and the Theory of Semiotic Mediation. The Theory of Semiotic Mediation is based on the instrumental approach and emphasizes the role of the teacher, while the Instrumental Approach to Mathematics Education emphasizes the co-development in the interactions between student and tool. The latter approach is rooted in cognitive ergonomics and has two directions, one evolving within Anthropological Theory of Didactics and another from Vergnaud's definition of schemes (Artigue & Trouche, 2021).

1.4 The Danish school system

The Danish education system mandates ten compulsory years of schooling, spanning primary education (0th grade–6th grade) to lower secondary education (7th–9th grade). Upper secondary education, inclusive of 10th to 12th grades, branches into academic (called *gymnasium*), vocational, and technical tracks (Ministry of Children and Education, 2018; Ministry of Children and Education 2021).

The national mathematical curriculum, which applies from primary school through upper secondary school, incorporates mathematical competencies as outlined in the KOM framework (Ministry of Children and Education, 2019; Ministry of Children and Education 2021) and provides general aims and guidance. Denmark maintains a free market for teaching materials, allowing teachers and schools the freedom to choose the materials they prefer to use (Danmarks Evalueringsinstitut [EVA] 2009). This also provides ample opportunities for research and collaboration with practitioners without compromising curricular required plans and materials.

1.3 Personal motivation

As a mathematics teacher, KOM has been my didactical anchor for teaching mathematics. However, taking a master's degree in didactics of mathematics became an eye opener for what MER has to offer practitioners. I often found that much of the existing literature on reasoning adopted an expert perspective on students' reasoning attempts (e.g., Duval, 2007; Harel & Sowder, 2007), evaluating students based on what their arguments lack, from an idealistic mathematical standpoint, rather than being curious about students' reasonings and acknowledging the learners' paths as valuable. The expert perspective simply did not resonate with the experiences I had as a teacher in primary and lower secondary school. The expert view was also present in the original version of RC, holding the ideal of proof and logic to be essential for the competency. However, the revised version of RC (Niss & Højgaard, 2019) articulated an inclusive view of reasoning and reasoning processes. I found that the revised version of RC presented an opportunity to approach reasoning from a student perspective, considering what students find evidential and how that emerges in their justification processes using DGAE. It has therefore been crucial for me to adopt and elaborate student perspectives in justification processes, and I hope to contribute to MER by giving insights on students reasoning as valuable in their own educational journey.

Another implication of my master's studies was that I became aware of the vast knowledge in the research field that has yet to disseminate into practice. As a way of impacting the field beyond my own classroom, I became an editor of digital resources for an educational publishing house, responsible for a mathematics platform for digital learning resources. One of my main tasks was integrating GeoGebra into the platform, which required collaboration with mathematicians, teachers, authors, and technicians. The experiences I gained in this position have been invaluable as I transitioned into my PhD project. GeoGebra proved to be more than a mathematical problem-solving tool, as it became a transformative environment for didactical design, enabling the creation of templates and resources that lets students explore mathematical concepts. Ultimately, for me this study has been an extension and a deep dive into aspects that were already at heart, designing mathematics educational resources well-informed by research.

1.4 STRUCTURE OF THE DISSERTATIONS

The dissertation is composed of six research papers and this kappa. The kappa introduces and explains the project, relates the results of the six papers, and provides additional results. It consists of eight chapters: 1) the introduction, 2) the theoretical foundation and perspectives 3) research questions, 4) the methodology, 5) the execution of the study, including a summary of the six papers, 6) design results, 7) analytical results, 8) theoretical developments, 9) the discussions and conclusion.

Chapters 6, 7, and 8 each contain a discussion of the results, and the final discussion in Chapter 9 revisits the methodology and discusses quality and overarching topics and concludes the study.

Appendix A and B contain sets of tasks used in classroom experiments, and Appendix C includes a list of abbreviations used in the kappa.

Chapter 2 begins with a discussion of the concept of theory, followed by a description of networking of theory as a research practice aligning with the research aim. It then presents the theoretical framework and foundation of the project, discussing the KOM framework, reasoning in school mathematics, and the use of tools in mathematics education, including the instrumental approach to mathematics education (IAME). Chapter 3, building on the theoretical framework, introduces and elaborates the projects three research questions.

Chapter 4 describes the methodology of the study, which is educational design research (DR), including how networking practices are intertwined in the processes and the use of theory in DR. Chapter 5 details how the DR methods have been executed and elaborates on the type of DR study, overview of the execution of the study, and provide contexts for classroom experiments, as well as methods for collecting data. The purpose is to clarify the working process and the associated decisions. Then, it provides an overview of publications related to the study, and it describes the six papers in relation to the study and its results.

Chapter 6 details the design processes and classroom experiments that led to the development of practical design principles, a microworld, and tasks. The experiments were conducted in three iterations of preparation, testing, and analysis.

Chapter 7 analyzes and discusses the practical implications of the relationships between students' tool use and RC, drawing on findings from papers, as well as additional analysis and results. Chapter 8 describes and discusses the theoretical developments throughout the study, presenting and proposing practical links of RC with the KOM framework and the IAME.

Chapter 9 discusses the methodological steps taken toward answering the research questions and the quality of the results. It also debates various practical aspects across the study, such as assessing RC in the context of tool use and how a particular tool within GeoGebra, the slider tool, can be beneficial for justifying algebraic structures. This leads to the conclusion of the dissertation, which summarizes the answers to the overall research questions.

2 THEORETICAL FOUNDATION AND PERSPECTIVES

This chapter introduces the theoretical considerations and approaches of the study as a whole and across papers and forms the basis for formulating the research questions presented below in Chapter 3. The chapter comprises two parts. The first part deals with fundamental questions and the perspective of scientific use of theory. It includes a discussion of the notion of *theory* and its use, and an overview of the networking approach. The second part elaborates on the theories applied in the study, as well as state of the art concerning the central aspects of the project: students' exercise of RC in justification processes and the use of tool.

2.1 PERSPECTIVES ON THE NOTION OF THEORY

"There is *no shared unique definition* of theory or theoretical approach among mathematics education researchers" (Bikner-Ahsbahs et al., 2014, sec. 2). Hence, it is crucial to provide a clear and comprehensive discussion of the concept of theory before employing the theoretical frameworks of this study.

Theorizing serves several purposes, and usually more than one is at play. In MER, Niss (2007) identifies several objectives of using theory. A theory can be used to explain phenomena, which is closely related to predicting phenomena, as the latter may depend on the former. Theory can be utilized to guide action or implementation to achieve a specific objective or to prevent scientific misconduct and criticism. Lastly, theory can provide a structured set of lenses to observe and interpret domains of the real world. Niss (2007), along with Schoenfeld (2007), signify the importance of this particular purpose:

> all empirical research is concerned wide and deeply grounded in (at times tacit but nevertheless strong) theoretical assumptions. Even the simplest observations or data gathering is conducted under the umbrella of either implicit or explicit theoretical assumptions, which shape the interpretation of the information that has been gathered. Failure to recognize this fact and to act appropriately on it can render research worthless or misleading. (p.70)

Mason and Waywood (1996) have put notions to this issue with their distinction between foreground and background theory. Foreground theory is an explicit form of hypothesizing in mathematics education that involves asking and answering questions. In MER most of the theoretical work falls within this category. Developed frameworks ore construct present an explicit hypothesis about what occurs, or can occur, under specific circumstances and can serve various functions, including description, explanation, prediction, and informing practice. Background theory, on the other hand, refers to implicit hypothesizing or beliefs that guide behavior. This includes the aims and goals of the research, the objects studied, the methods used, and the perceived situation, all of which are shaped by a philosophical stance. According to Mason and Waywood (1996), research in mathematics education is based on a background theory of mathematics education, which does not become a foreground theory even if the hypothesis becomes explicit. This means that theorizing, such as framing questions, collecting data, and analyzing results are determined and constructed by the discourse and philosophical stance of the background theory, *but are not used with an explicit aim*. Examples of such theories are postmodernism, phenomenology, or constructivism. The background theory hence provides the conditions for the structure of the research, but it is not itself a theory generated within MER. The distinction captures the fact that MER theories are traditionally inspired by other fields (e.g., psychology, general education, and mathematics), making the distinction between background and foreground theories paramount when describing the core of a theory. More recently, however, MER has shifted toward theory building within the research field, rather than relying on theories borrowed from other fields (Lesh et al., 2014). These theories do not necessarily build on a background theory outside MER. One can consider the grounded theory approach (Vollstedt & Rezat, 2019), where theory emerges from an empirical discourse rather than a theoretical one. Considering the growing trend of theory building within MER brings perspective to the remark of Bikner-Ahsbahs et al. (2014), that if background and foreground theories are considered "relative distinctions rather than an absolute classification, they can help to distinguish different views on theories" (p. 6). In paper 6, we acknowledge that theory from inside MER can have elements that act as background theories in research practices. Concurrently, we challenge whether a relative distinction is purposeful, as it may blur the fact that a background theory resides outside of MER and has a larger scope. Furthermore, we identify cases of the use of the relative distinction, where theory from inside of MER acted as background in certain situations and as foreground in other situations. To accommodate the need to characterize such elements and this dynamic use of theory, we suggested that theories from inside MER that act as background theories are at least referred to as *background theories inside MER* or *framing theories*.

As already noted, the definition of theory is not agreed upon in MER. Theory generally consists of a core, empirical components, and an application area (Prediger, Bikner-Ahsbahs, et al., 2008) Bikner-Ahsbahs et al. (2014) suggest that theories should be viewed dynamically, as they are always in a *state of flux*. This is due to the inherent dialectic of theorizing: theories guide research practices and are in return influenced by them or even become the aim of research practices. This contrasts with a static view of theory as a finished analytic tool that organizes and systematizes parts of the real world. Although agreeing that theories are in a *state of flux*, Niss (2019) maintains that it "…does not mean that the definitions of the concepts are as well", and that "if we refuse to offer definitions of these terms, we end up taking them for granted according to our private understanding, and we don't know what we are talking about!" (Niss, personal communication, May 9, 2019). Hence, he challenges the overly broad definitions of theory, such as, e.g., in the JRME editorial by Cai et al. (2019):

> In this editorial, we use the term theoretical framework broadly (similar to the treatment of conceptual frameworks by Eisenhart, 1991, and Lester, 2005) to encompass the set of assumptions, theories, hypotheses, and claims (as well as the relationship between them) that guide a researcher's thinking about the phenomenon being studied. (p. 219)

Radford (2008) suggests a triplet set (*P, M, Q*), where "theory can be seen as a way of producing understandings and ways of action based on:

- A system, *P*, of *basic principles*, which includes implicit views and explicit statements that delineate the frontier of what will be the universe of discourse and the adopted research perspective.
- A *methodology*, *M*, which includes techniques of data collection and data interpretation as supported by *P*.
- A set, *Q*, of paradigmatic *research questions* (templates or schemas that generate specific questions as new interpretations arise or as the principles are deepened, expanded or modified). (p. 320)

Radford (2012) later extended the definition to encompass *results* (*R*), which drive the development of *P*, *M* and *Q*: "There is indeed a dialectical relationship among the various components of a theory. The dialectical relationship is mediated by the *results* that a theory produces" (p. 5). Radford's definition addresses clearly 'the core' of a theory in terms of the system of *P* and 'the application' by M and Q. However, 'the empirical component' is a connotation in the system of *P*.[2]

Radford does contemplate how facts of objects appear throughout the history of scientific investigation, and he outlines two positions. In the first position, "the fact refers to general principles; the fact is a particularization of the general" (Radford, 2012, p. 3). In the second position, "the fact generates the principle through an inductive process" (Radford, 2012, p. 3). He concludes:

[2] Examples of object and phenomena in *principles* are found in Radford (2018)

“The understanding of the phenomena under investigation can only be achieved against the background of general principles” (Radford, 2018, p. 4). From his conclusion, we may understand that principles are general facts about objects. An example, formulated by Radford, is a principle of constructivism about the phenomenon of knowledge appropriation: “*knowledge is not passively received but built up by the cognizing*” (Radford, 2012, p. 4). Despite Radford’s elaboration on objects of theory, Bikner-Ahsbahs and Prediger (2014), in their comparison of five different theoretical approaches, found a need to add “key constructs” to *P* in the triplet *PMQ*. Key constructs is a term for the objects and phenomena under study, and it underlines a need for stronger clarification of the empirical component than Radford’s definition offers. Thus, the implicitness of objects in Radford’s definition is problematic since the ‘objects’ of research are essential when considering which theories are comparable and compatible, and how. As Niss (2019, personal communication, May 9, 2019) argues, a theory must be a theory of something, and Niss and Jankvist (2022) suggest a definition that centralizes the objects and phenomena of a theory:

> A *theory* is a theory *of something*, i.e., it deals with certain sorts of *objects* and *phenomena* and includes terms for these. Its purpose is to produce *corroborated claims* about these objects and phenomena, typically in response to *questions* posed about them. These claims are generated by some *means*, on some *grounds*, involving some fundamental *methodology/ies*. (p.17)

There are notable similarities between Radford and Niss and Jankvist. Niss and Jankvist’s definition includes “questions” similar to Radford’s *Q*; Niss’ and Jankvist’s “means” and “methodology/ies” are similar to Radford’s *M*; and finally, Niss’ and Jankvist’s “corroborated claims” and “grounds” are similar to Radford’s *P*.

The main difference between the two definitions is their central focus. Radford’s definition focuses on the set of principles. P is the first-mentioned element and reoccurs in the description of both *M* and *Q*. Radford’s concern with the dialectic surrounding phenomena – the system of *P* – gives strength to an elaboration of a background theor, as theory is rooted in a systematized discourse. Radford has so far not elaborated on the scope of his definition. It is unclear whether he differentiates theory elements that can be perceived as foreground theory such as theoretical framework, theoretical approach, and theoretical construct, which leaves an undefined cluster of theoretical notions.

Niss and Jankvist’s (2022) definition focuses on objects and phenomena and the production of predictive or explanatory claims about such. Consequently, the values and norms of the theory are implicit (Prediger, Bikner-Ahsbahs, et al., 2008) in the corroborated claims and grounds. Niss (2007) and Niss and Jankvist (2022) also stress that only a few actual theories exist in MER:

> Although we would, in fact, grant the label “theory” to some constructions in mathematics education—e.g., Brousseau‘s (1997) theory of didactical situations, or Chevellard‘s (2019) anthropological theory of the didactic (ATD), and the APOS theory developed by Dubinsky (1991) [...] At any rate, the number of theories in mathematics education is, at best, extremely small. (p. 5).

Given that Mason and Waywood’s (1996) definition of background theory includes the aims and goals of the research, the objects studied, the methods used, and the perceived situation, Niss and Jankvist’s (2022) definition reflects this sense of theory. Contrary to Radford, they do consider the aforementioned cluster of theoretical notions that reflect foreground theory. They describe a *theoretical perspective* as one or more theoretical approaches that are part of the solution to the research problem. The following notions of theory are examples of the extent of different theoretical perspectives. A *theoretical construct* is a concept introduced by way of a definition. It can rest on certain assumptions or hypotheses and can involve certain claims. It can be a singular construct such as *sociomathematical norms* (Yackel & Cobb, 1996) or a distinction such as Tall and Vinner’s (1981) *concept image* and *concept definition*. A collection of one or more theoretical constructs is considered a theoretical framework that:

> frames—i.e., provides the foundation for—the conceptualisation, design or carrying out of the study, including its interpretations, analyses or inferences. The elements of a theoretical framework do not have to be linked so as to form a full-fledged theory. In fact, the framework does not even have to be coherent or exhaustive but may take the shape of *bricolage* (Cobb, 2007; Gravemeijer, 1994) of singular theoretical constructs. (Niss & Jankvist, 2022, p. 18)

Finally, a *theoretical approach* is a conceptual and theoretical investigation of the research problem, usually by incorporating or developing some theoretical constructs.

Niss and Jankvist (2022) also draw out three different target levels or grain sizes of theoretical approaches: local, medium, and global. Global corresponds to their definition of a theory and are hence scarce in MER. Medium deals with “a generic set of topics or issues across several domains” (p. 19), while local deals with a specific topic or issue.

In the closing of this part, I would like to underline that the intention is not to take sides or favor one tradition over another. That said, to be consistent in terms and approach I will adhere to Niss and Jankvist’s (2022) definition of theory and notions of theory, as they offer a more elaborate system of notions. Prediger, Bikner-Ahsbahs, et al. (2008) appeal for a broad definition to not exclude any *theories*. Indeed, Niss and Jankvist’s (2022) approach excludes most theoretical approaches in MER as *a theory*. However, in the cluster of notions of theory, the theoretical approach is the most inclusive notion, where only empirical approaches are excluded.

2.1.1 Linking theory

Lerman (2006) pleads for considering the diversity of theoretical approaches in MER as a source of richness that is necessary to grasp complexity. Prediger, Bikner-Ahsbahs, et al. (2008) argue that the "richness of plurality can only become fruitful when different approaches and traditions come into interaction" (p. 169). Fruitful interaction of theoretical approaches can advance different purposes, such as better understanding of theories, capitalizing on collective research results, obtaining coherency in MER, controlling the excessive growth theories, improving teaching and learning in mathematics education, and finally, guiding design research (Prediger, Bikner-Ahsbahs, et al., 2008).

Networking of theories (NT) progresses fruitful interaction between different theoretical approaches. It has grown out of the thematic working group "Theoretical perspectives and approaches in mathematics education research" at the Congress of European Research in Mathematics Education (CERME) (Kidron et al., 2018). The group confronts the diversity of MER theories, both as a challenge for the research community but also as a richness that can be a starting point for a development. Subsequently, NT aims to produce understanding and connections between the myriad of theories. Prediger, Bikner-Ahsbahs, et al. (2008) have systematized and termed strategies for relating theories, which they collectively call *connecting strategies* (Bikner-Ahsbahs, 2016; Prediger, Arzarello, et al., 2008). The strategies draw out a one-dimensional scale where strategies are placed according to the degree to which they integrate theoretical approaches (see Figure 1). The extreme strategies of *ignoring other theories* and *unifying globally* are not considered as NT. In the first case, as implied, no connections are made, resulting in the cultural isolation of theoretical approaches. The latter speaks to the idea of one global theory of mathematics education, which in practice would result in ignoring conflicting theories and devaluating the richness of diversity that Lerman (2006) pleads for. Ultimately, unifying globally acts as a virtual extreme. As depicted in Figure 1, the intermediate strategies are defined as networking strategies that "are those connecting strategies that respect on the one hand the pluralism and/or modularity of autonomous theoretical approaches but are on the other hand concerned with reducing the unconnected multiplicity of theoretical approaches in the scientific discipline" (Prediger, Bikner-Ahsbahs, et al., 2008, p. 17). Often more than one strategy is needed to reach the aim of a given process of NT. For instance, *understanding others* and *making understandable* will be the first step to obtain any higher degree connection. The strategies are presented in pairs that serve the same aim.

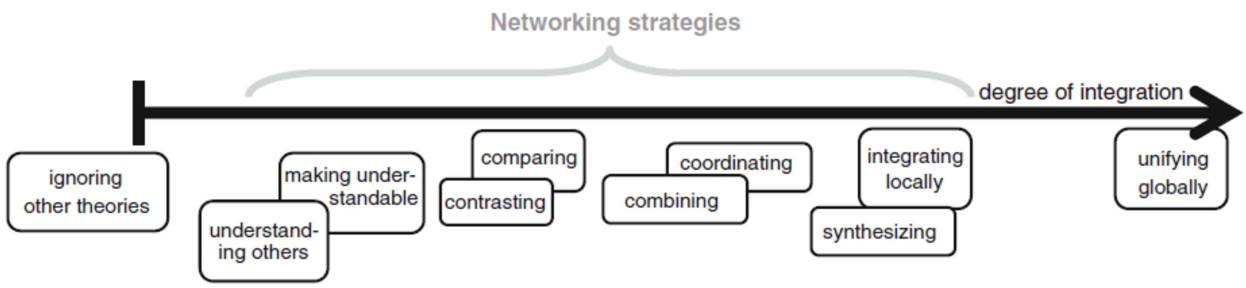


Figure 1 – Prediger, Bikner-Ahsbahs, and Arzarello (2008). Landscape of strategies for connecting theoretical approaches

Understanding others and *making own theories* understandable is essential for in networking practices. The strategies goes beyond understanding definitions and the hierarchy of terms and methodology to uncover both explicit and implicit assumptions (Prediger, Bikner-Ahsbahs, et al., 2008). Understanding other theoretical approaches can be an aim in itself, and conversely, practicing NT can contribute to a deeper understanding of both one's own and other theories, which is why these two strategies are an implicit permanent aim of any NT practice.

Comparing seeking commonalities between theoretical approaches and *contrasting* points to differences. Both can take on one or more of three theoretical aims: furthering the understanding of the investigated approaches, positioning a theory (as valuable) in the field of theories in MER, and/or describing a rational for the choice of theoretical approaches on a meta-level. Comparing and contrasting can, like the previous strategies, be a step toward further integration.

In contrast to the former strategies, *coordinating* and *combining* has the empirical aim of understanding a phenomenon or a piece of data (Prediger, Bikner-Ahsbahs, et al., 2008). These strategies are relevant in triangulating a phenomenon, where two theoretical approaches can give a richer and deeper insight and to capture phenomena that have inter-relational variables (e.g., students' use of digital tools and their RC) which cannot be captured with a single theoretical approach. They often result in a conceptual framework but do not necessarily present as completely coherent. Coordinating aims at coherency between well-fitting elements from different approaches, whereas combining is a juxtaposition of approaches and has a less ambitious aim for the coherency. To ensure coherency when coordinating, theory elements should be carefully analyzed, and only theories with compatible cores considered coherent (Prediger, Bikner-Ahsbahs, et al., 2008).

Synthesizing and *integrating* aims at theory development. These strategies are a continuation of coordinating, moving beyond the understanding of empirical phenomena to theory building with coherent approaches. The two strategies differ in the status of the theoretical approaches in play.

Synthesizing demands a symmetry between the approaches, as when “two (or more) equally stable theories are taken and connected in such a way that a new theory evolves” (Prediger, Bikner-Ahsbahs, et al., 2008, p. 173). Integrating is when there is a difference in scope between the approaches, and elements of only one approach are integrated into the other.

2.1.2 Methods and practices for NT

Each of the strategies can be carried out using different methods, which will spring from the approach, focus, concepts and methods in play, and the aims of the networking. Hence, there are a diversity of methods and methodology within each strategy (Prediger, Bikner-Ahsbahs, et al., 2008). However, some strategies can be considered particularly relevant to pair with specific methodologies. Coordinating and combining fit the aims of design research (diSessa & Cobb, 2004) for understanding an empirical case and developing theories (Prediger et al., 2008; Prediger, 2019), for example, presenting a case where strategy coordinating is embedded in a design research study to strengthen the empirical analysis and the development of design principles (Van den Akker, 1999).

Networking practice involve close attention to both the theoretical approaches. The higher integration, the more carefully one must consider coherency and compatibility of the background of the theoretical approach and the different elements of the theory (Prediger, Bikner-Ahsbahs, et al., 2008). Radford (2008), in continuation of his definition of theory, suggests comparing theories by the set (P, M, Q) to reveal commonalities, distinctions, and compatibility. He argues that connections can be made between the same elements, e.g., between the principles of each theoretical approach, or across elements, e.g., between the principles of one theory and the methodology of another. Radford (2008) also conjectures that: “theories are more likely to be connected if their theoretical principles (or at least some of them) are ‘close’ to each other” (p. 323).

Niss and Jankvist (2022) suggest a graph-theoretical metaphor in which a set of nodes are theoretical entities that may be linked by edges to form a network. The links can be different, and they argue that: “A fundamental issue for linking two theoretical entities is whether these represent two different ways of dealing with the same object(s) or phenomena, or whether they deal with different objects or phenomena” (p. 32). Like Radford (2008), they stress that the purpose of connecting theories is fundamental to the nature of the connection. In their considerations of linking KOM to other theoretical approaches, they reflect on two purposes. *Mutual fertilization* by linking a theoretical approach to KOM, where both elements are enriched with perspectives they did not contain on their own and *a methodological means to uncover a phenomenon* of which each theoretical approach has methods to capture different aspects. These goals fall within the strategies of coordinating and combining, which Niss and Jankvist argue to be the highest degree of integration that can be done with the KOM framework.

To make connections between theoretical approaches, Bakker (2016) suggests the use of boundary objects to assist in crossing the cultural boundary between theoretical approaches. Boundary objects must be

> both plastic enough to adapt to local needs and the constraints of the several parties employing them, yet robust enough to maintain a common identity across sites. They are weakly structured in common use, and become strongly structured in individual site use. (Star & Griesemer, 1989, p. 393)

Bakker (2016) elaborates on how similarities can be drawn between the approach of boundary crossing and NT, as both practices consider how “people seek to make connections between practices or praxeologies that have different origins and purposes” (p. 271). To cross boundaries, they must become permeable through communicative connections and efforts of translation. In this optic, boundary objects are artifacts with a bridging function between practices so that initial boundaries between research practices become permeable (Bakker, 2016). In the context of this study, two separate research practices are in play, the study of students’ RC and the study of students’ use of tools. A theoretical boundary object can assist in making mutually fertilizing connections between the two practices.

2.2 OVERVIEW OF THE PROJECT AND THEORETICAL FRAMING

Moving on from fundamental questions of theory, I now present the concrete theoretical framework and approaches of the study and elaborate the state of the art of central aspects.

To provide an overview of the theoretical elements in the project, I use the pentahedron of Zbiek et al. (2007) (see Figure 2), which signifies how digital technology influences various aspects of mathematical learning among students. It contains the nodes student, mathematical content, mathematical representation, mathematical activity, and digital tool. Lines illustrate the interrelations between all nodes, and as the digital tool mediates activity between all nodes, the dotted lines illustrate that the digital tool influences not only each node but also each relationship (Zbiek et al., 2007). The pentahedron hence illustrates a closed system with the student as actor and the tool as mediator between nodes. Each node of the project is specified in Figure 3. As mentioned in the introduction, the digital tool used in the study is GeoGebra, and the mathematical activity is

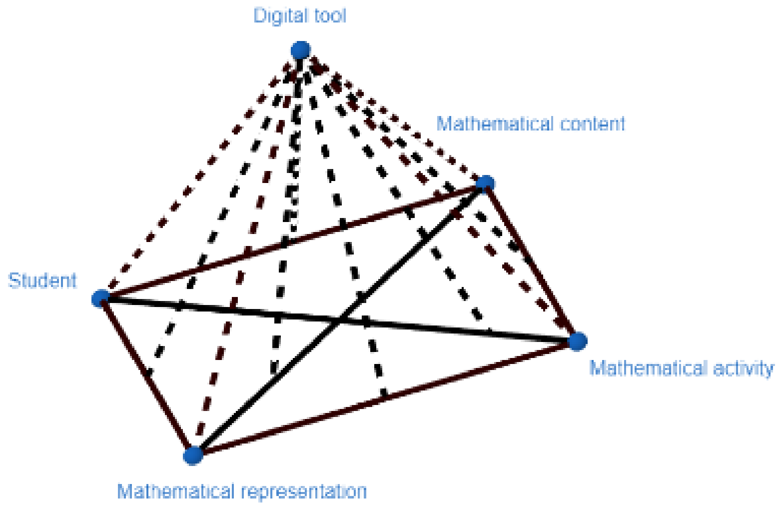


Figure 2 – Replication of Zbiek et al. (2007) pentahedron with nodes: student, mathematical representation, mathematical content, mathematical activity, and the digital tools and relations among them. Dotted lines represent the influence of the digital tools on nodes and relations between nodes

the exercise of RC in justification and concern students in lower secondary schools. The mathematical content is variable as a general number in ordered pairs represented symbolically and graphically in the coordinate plane. This allows for using the algebra and graphic views in GeoGebra.

I utilize different perspectives to capture the different relationships. KOM (Niss & Højgaard, 2019) conceptualize the relationship *student-mathematical activity* and the IAME conceptualize the

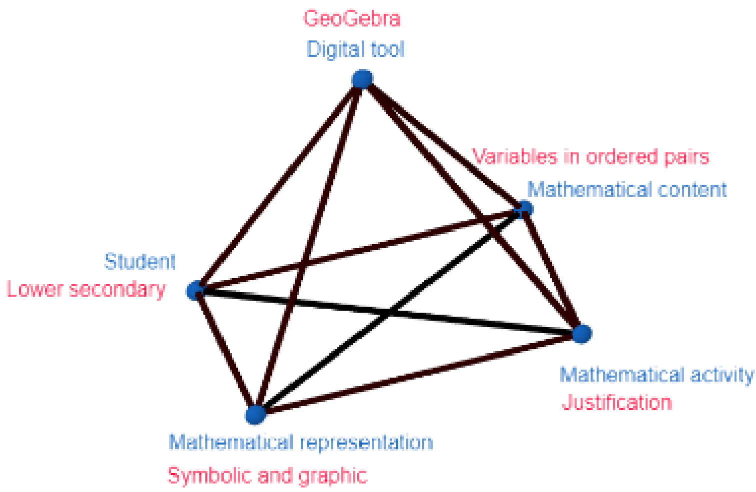


Figure 3 – The nodes of pentahedron concretized with respect to the study

relationships *student-digital* and *tool-mathematical content*.

The illustrations in Figure 4 show that the common node is the student but otherwise, there are no overlaps between KOM and the Instrumental Approach. To bridge the two, the lack of overlap must be addressed in the theory development of the study. What also appears is that the node mathematical representation and its relationships are not theorized and hence mark the study's limits. However, the node is relevant and is elaborated in this chapter 6, with regard to DGAE. The

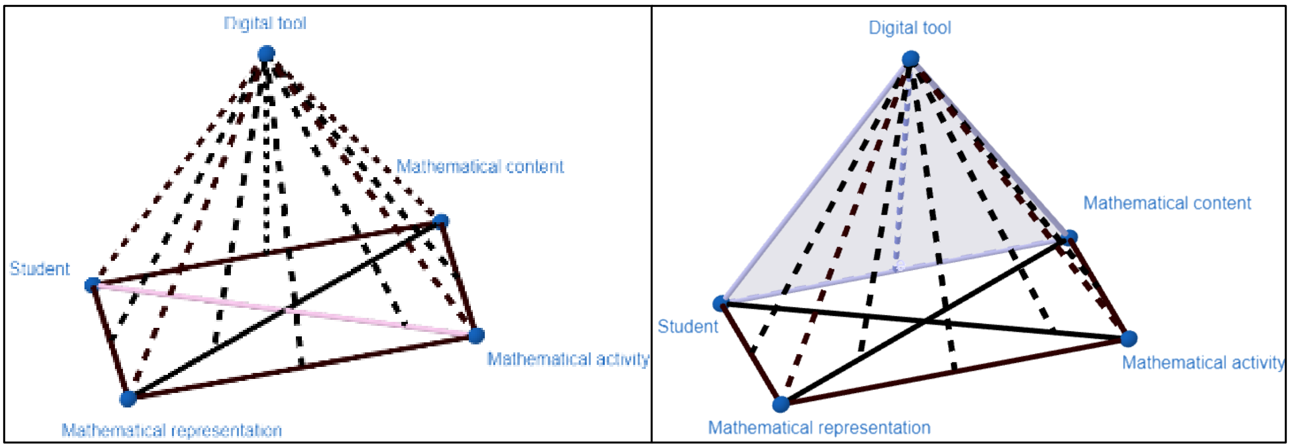


Figure 4ab – a) The KOM framework conceptualizes the relationship *student-mathematical activity*. b) The Instrumental Approach to Mathematics Education captures the relationships *student-digital* and *tool-mathematical content*

mathematical content node concerning variables in ordered pairs is elaborated in Chapter 6 for the context of designing tasks. Finally, even though the pentahedron is represented as a closed system, it is part of a more extensive, surrounding system containing relationships to, e.g., other students, the teacher, and institutions, which are also not theorized within the study.

In the follow section 2.3, describe the KOM framework. Section 2.4, position the study within reasoning and justification in MER and section 2.5 discuss GeoGebra within the traditions of both CAS and DGE, and describe GeoGebra's interface and how variables are represented and used in GeoGebra. Finally, section 2.6 discuss and explain the IAME.

2.3 The KOM-framework

The competency description framework KOM describes mathematical mastery from the perspective of competence. It was first introduced in the KOM report (Niss & Jensen, 2002), which presented the results of the KOM project, developed by an extensive group of people lead by Niss, and organized and founded by the Council of Science Education and the Ministry of Education in Denmark (Niss & Højgaard, 2011). Since then, it has been translated into English in 2011 and revised in a more condensed form aimed at an international audience in 2019 (Niss and Højgaard (2019).

In KOM, a mathematical competency is defined as "someone's insightful readiness to act appropriately in response to a specific sort of mathematical challenge in given situations" (Niss & Højgaard, 2019, p. 6). In total, there are eight distinct but related competencies, which are illustrated in Figure 5.

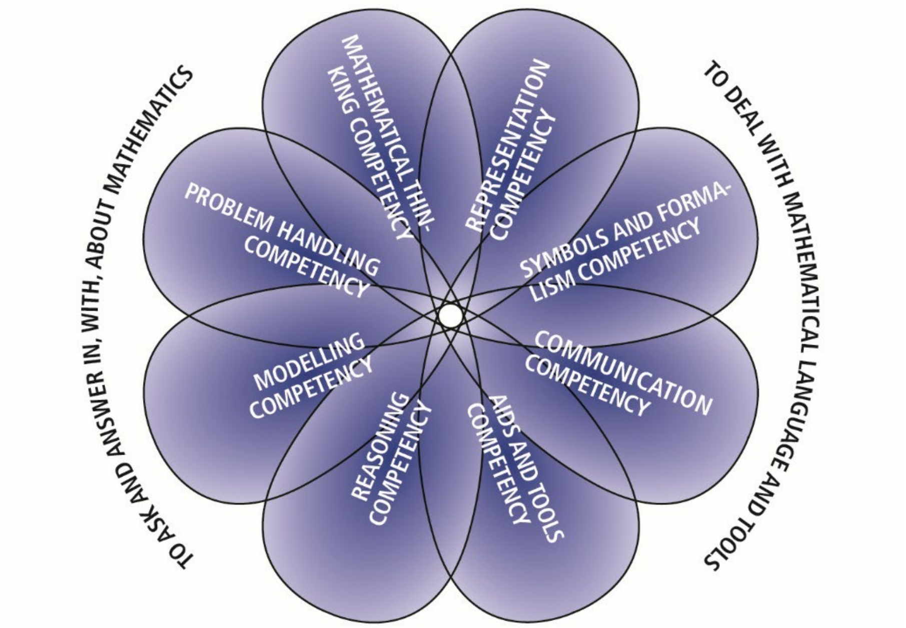


Figure 5 – The KOM flower depicting the eight mathematics competencies

The competencies are divided into two categories: *asking and answering in, with, and about mathematics* and *dealing with mathematical language and tools*. I focus on the RC in the first category, which I will describe further in the next section. However, it is important to note that tackling a mathematical challenge requires multiple competencies. While one competency may be in focus in a particular educational or research setting, other competencies will be relevant. For example, in this study, using a digital tool requires the students' aids and tools competency, as well as representation competency and symbol and formalism competency to handle the multiple representations of GeoGebra. Therefore, after describing the RC, I also account for those.

The mastery of analyzing or producing mathematical arguments require RC. Arguments can be presented orally or in writing and can take various forms, such as exemplifying and deductive or formal proofs. An argument is a series of statements connected by inference and used to support mathematical claims or solutions to mathematical problems (Niss & Højgaard, 2011, 2019).

A competency can be described and assessed according to its *degree of coverage*, *radius of action*, and *technical level*. Essentially, this captures that competencies are related to form, situations, and complexity:

- The *degree of coverage* concerns the aspects of the competency. For the RC, these aspects are to actively participate in oral and written reasoning for mathematical claims and critically analyze and assess existing justifications and justifications put forward by others. The forms of reasoning can be placed on a wide spectrum, from providing examples to rigorous proof.
- The *radius of action* is the variety of different contexts, mathematical or general, in which the competency can be used. Contexts can be mathematical domains, different social situations, or different mathematical situations.
- The *technical level* concerns the sophistication of concepts, results, theories and methods used within the competency.

What activities fall within a given competency can in some cases be unclear. In principle, the ability to carry out pure routine operations may fall within the RC since it involves justifying the results of calculations. However, what one person may consider a routine operation, another person may view as an insurmountable problem. Therefore operations is included under "the competency dealing with mathematical symbols and formalisms while being able to activate the operation belongs under the RC if this activation demands creativity, analysis, or overview" (Niss & Højgaard, 2011, p. 61). The symbol and formalism competency is, hence, related to the exercise of RC and in particular the technical level.

Likewise, the problem handling competency is particularly relevant, as problem-solving constitutes a context for the student's exercise of RC (Niss & Højgaard, 2019). The process of obtaining an answer to a mathematical problem is core in problem handling competency as it involves posing and solving mathematical problems by devising and implementing problem-solving strategies. It is closely related to the RC, as it concerns justifying strategies and solutions.

When it comes to using a DGAE, there are certain competencies that are relevant. The multiple representations in a DGAE require students to translate or interpret between these representations. This requires representation competency, as well as understanding the strengths and weaknesses of different representations (Niss & Højgaard, 2019).

Finally, the tools and aids competency is relevant, as it encompasses the constructive use of tools in mathematical work. This also involves considering the affordances and limitations of different tools. Niss and Højgaard (2019) underline the diverse physical properties of tools, which may not necessarily have direct implications in mathematical contexts. This presents challenges that require thoughtful consideration when integrating tools into mathematical situations.

The development of a person's mathematical competence is achieved through active participation in various mathematical situations. Competencies can be assessed over time or compared between individuals as progress in the three dimensions (Niss & Højgaard, 2011, 2019), but it is important to remember that competency is context-specific. This study aims to examine the processes of justification in the context of RC, rather than evaluating progress of student's RC. Verbs such as *activating*, *implementing*, *displaying*, and *expressing* can be used to describe a student's use of their competency. To emphasize the students as learners, I use the verb *exercise*. Thus, students are exercising their RC in their "enactment of mathematical activities and processes" (Niss & Højgaard, 2019, p. 3).

2.4 REASONING AND JUSTIFICATION IN SCHOOL MATHEMATICS

The terminology of the reasoning literature in MER presents a diversity of perspectives and definitions. This diversity necessitates defining the terminology used in this study and describing how it situates the study within the broader literature on reasoning in mathematics education. Across the literature, central terms like *reasoning*, *proof*, *argumentation*, and *justification* may overlap or even be used interchangeably, but they can also be looked at from specific perspectives.

The study focus on the process of justification, however, to position the study I will first consider reasoning, proof, and argumentation in mathematics education.

Since RC is the competency in focus, I use *reasoning* as an overarching term in the remainder. Reasoning encompasses both the product and the process by which it comes to be, meaning that an argument is a product of argumentation, a proof is a product of proving, and a justification is a product of justifying/justification.

Reasoning can have different functions. All types of reasoning processes have in common that they aim to change the epistemic value of a statement (Duval, 2007), such as a conjecture, hypothesis, or theorem. The processes differ in how this change is obtained (e.g., through induction or deduction), the grounds on which they are based, and the types of claims they are relevant to.

Hanna (2000) provides a list of the different functions of reasoning and proof in mathematics:

> "• verification (concerned with the truth of a statement)
>
> • explanation (providing insight into why it is true)
>
> • systematisation (the organisation of various results into a deductive system of axioms, major concepts and theorems)
>
> • discovery (the discovery or invention of new results)
>
> • communication (the transmission of mathematical knowledge)

• construction of an empirical theory

• exploration of the meaning of a definition or the consequences of an assumption

• incorporation of a well-known fact into a new framework and thus viewing it from a fresh perspective” (p. 8)

In an educational setting, some functions hold relevance, and according to Hanna (2000), reasoning and proof should be at least explanatory to have educational significance.

2.4.1 Argumentation and proof

In KOM, arguments are considered chains of statements linked by inferences to justify mathematical claims (Niss & Højgaard, 2019). Hence, argumentation takes place in different forms of reasoning.

An argument can be considered from a structural perspective. Toulmin (2003) poses a geometric model (see Figure 6), rooted in jurisdiction and thus developed to investigate what constitutes a valid argument from an epistemological and psychological stance. His model schematizes the basic elements of an argument: a claim along with a qualifier, data, and a warrant. The claim is a statement or conjecture, and its epistemic value is indicated by the qualifier (e.g., false, possible, likely, more likely, or true), an expression of the probability of the claim. The qualifier is established based on the data that support it (the evidence) and the warrant, which includes inferences connecting the data to the claim. The role of the warrant is “to show that, taking this data as a starting point, the step to the original claim or conclusion is an appropriate and legitimate one” (Toulmin, 2003, p. 91). Extended elements are the backing, which can provide support for the warrant, and the rebuttal, which can include limitations of the claim or counterarguments.

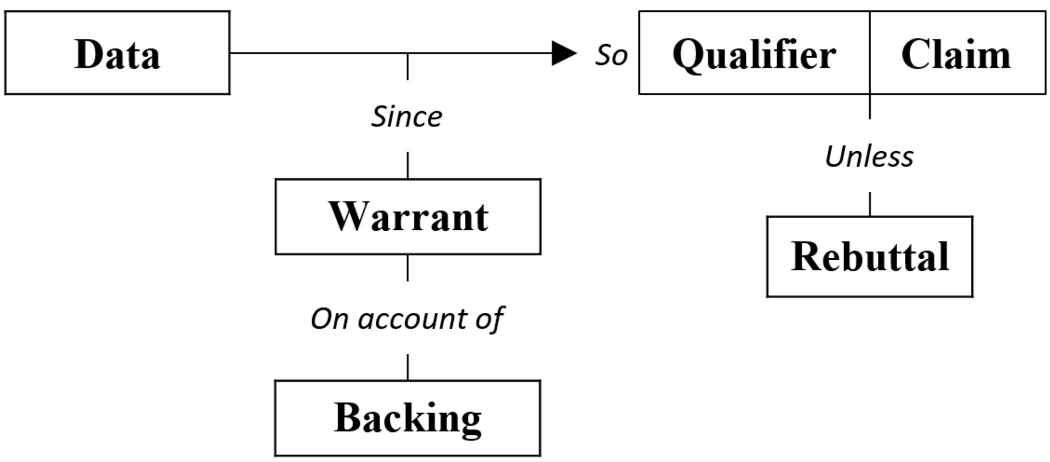


Figure 6 – Elements of Toulmin’s (2003) argumentation model

Another structural perspective, this one rooted in logic, is that of Peirce (Cohen, 1933) who describes three basic inference modes: deductive, inductive, and abductive. These are commonly discussed in the literature on reasoning and proof.

In Toulmin's argumentation model, (Jeannotte & Kieran, 2017). Deductive reasoning involves arriving at "new information derived from a set of premises via a chain of deductive inferences" (Harel & Weber, 2018, p. 1). The deductive structure is crucial in rigorous proofs, with proof being a particular kind of argument derived from assumptions and propositions endorsed by the mathematical community (Weber et al., 2014).

Induction infers a warrant from the data and a claim about the data (Jeannotte & Kieran, 2017). It is related to using examples to provide validity, or generalizing based on examples, including instances where a student relies on examples or mental images to verify the validity of an argument (Manouchehri & Sriraman, 2018).

Abduction can take two forms. It either has a structure that infers data from the claim and the warrant, or one that infers data and warrants from the claim (Eco, 1986; Jeannotte & Kieran, 2017; Pedemonte & Reid, 2011). Abduction is related to the explorative processes of reasoning, e.g., to discover a pattern or infer a rule, and it is often part of both the deductive and inductive processes.

2.4.2 Justification and justifying

While argumentation and proving have been extensively researched in MER, justification has received less attention (A. J. Stylianides & Stylianides, 2022). In KOM, justification is a general term for the argumentation for a mathematical claim. In this study, however, I follow a more narrow understanding of the term, considering justification particular to a problem-solving process, e.g., when students are asked to explain and warrant the answer for a given problem. Bieda and Staples (2020) define mathematical justification as "the process of supporting your mathematical claims and choices when solving problems or explaining why your claim or answer makes sense" (p. 103). It should be noted that some authors use the term *reasoning* correspondingly. For example, Lither (2008) defines reasoning as

> the line of thought adopted to produce assertions and reach conclusions in task solving. It is not necessarily based on formal logic, thus not restricted to proof, and may even be incorrect as long as there are some kinds of sensible (to the reasoner) reasons backing it. (Lithner, 2008, p. 257).

However, as previously described, I use reasoning as a collective term for a range of forms of reasoning. What is essens of justification is the context of problem solving, and that the epistemic value is assigned by the reasoner rather than the general mathematics community. Justification processes are not necessarily linear, as "students can take various paths in making sense of the claim, revising their stance on the truth value of the claim, or settling on a statement that is acceptable to their peers" (Lesseig & Lepak, 2022, p. 96). Early studies have elaborated on the social perspectives that influence what a student might conceive as convincing. Yackel and Cobb (1996), Wood (1999), and Wood et al. (2006) explored the relationship between explanation and justification, investigating the nature of students' responses, revealing sociomathematical norms guiding mathematical activity in the classroom. Simon and Blume (1996) viewed justifications as the responses students offered when asked to provide mathematical evidence, and they explored criteria for acceptable justifications within the teacher community. Dreyfus (1999) distinguished between descriptive and justificative modes of thinking, emphasizing the role of the activity in the classroom community. Common to these studies is the emphasis on the classroom culture to determine what constitutes a justification. From that perspective, what counts as a sufficient justification is cultivated by the teacher and students in the classroom, and to a lesser extent, the general mathematics community. This does not mean that the justification cannot rely on mathematical theory or lack deductive steps. However, if that is the case, it is a condition cultivated in the classroom or by the students participating in the process. Jeannotte and Kieran (2017) consider justifications to be validating processes of narratives that "by searching for data, warrant, and backing, allow for modifying the epistemic value of a narrative" (p. 12) and elaborate thus:

> The elements supporting the process are constrained by meta-discursive rules within a certain community. For example, the change from likely to true has to be based on a deductive structure. On the other hand, in changing from likely to more likely, some meta-rules constrain the process, but a deductive structure is not necessary (Jeannotte & Kieran, 2017, p. 12).

Meta-discursive rules govern how we discuss and reflect on our own discourse (Sfard, 2008). It is fundamental to determine the meta-discursive rules in a particular community if to infer the epistemic value of a claim. This is not the approach taken in this study, however. It can be assumed that in many lower secondary mathematics classrooms, deductive structured arguments are not (yet) developed as a practice, and a deductive step cannot be a meta-discursive rule related to the epistemic value *true*, as it would be in higher education. From the student's perspective, obtaining the epistemic value *true* is rather based on inferences from the *convictions* of the participating students. G. J. Stylianides (2008) suggests two "non-proof" arguments. The first is empirical justification, where the solver checks a proper subset of possible cases. The second is *rationale*, meaning if an argument "does not make explicit reference to some key accepted truths that it uses (in the context of a particular community where these truths can be considered as key), or if it uses statements that do not belong to the set of accepted truths of a particular community" (G. J.

Stylianides, 2008, p. 12). Empirical justification and rationales I consider likely to occur in justification processes. Although some studies confront students' reliance on empirical justification as a limitation to be overcome (e.g., Duval, 2007; Harel & Sowder, 2007; G. J. Stylianides & Stylianides, 2009), other studies highlight the significant role of examples in the justification process. Studies such as Pedemonte and Buchbinder (2011), Zazkis et al. (2008), and Knuth et al. (2019) demonstrate how testing specific examples enables students to form conjectures and explore the boundaries of generalizations. Importantly, students' empirical investigations have the potential to unveil mathematical relationships and structures that foster a deeper comprehension of the underlying concepts. Indeed, justification can promote understanding among engaged participants (Staples et al., 2012). However, students may accept justifications without grasping underlying mathematical concepts (Lesseig & Lepak, 2022). In this sense, Lithner's (2008) distinction between imitative and creative reasoning emphasizes the importance of students' engagement with sense-making in reasoning processes such as justification. Imitative reasoning is anchored in either memorized solutions, the enactment of algorithms, or authorities such as technology, peers, or teacher guidance. Creative reasoning involves novelty in the argument of the justifier in support of the plausibility of a claim (the epistemic value, likely), and it is anchored in intrinsic mathematical properties (elaborated in 6.1.1).

2.5 DIGITAL TOOLS IN MER

Digital tools have garnered interest in MER for decades. One of the pioneers in this field was Papert (1980), who advocated for the use of digital tools as a means of constructing knowledge and encouraging critical thinking. This optimism toward the potential of digital tools in mathematics education was also reflected in the first ICMI study on the topic (Churchhouse et al., 1986). However, it has since been recognized that the educational value of digital tools is not inherent, but must be promoted within the educational context (Drijvers et al., 2016), as the same digital tool can both enhance and replace mathematical competency and capacities (Niss, 2016).

Digital tools can be utilized to delegate processes, such as calculations or drawing figures, that are tedious, difficult, or prone to producing inaccurate results (Buchberger, 1990; Hoyles, 2018). Outsourcing processes can free up students' time to focus on other activities that develop conceptual knowledge, such as investigating the results or the processes themselves (e.g., Gyöngyösi et al., 2011; Segal et al., 2016). However, outsourcing can also *black box* (Buchberger, 1990) the processes, making them inaccessible for students to understand and causing conceptual misunderstandings when interpreting results (Jankvist & Misfeldt, 2015). Therefore, the use of digital tools in mathematics education should be evaluated based on their educational value so that it supports students' conceptual development (Artigue, 2002).

GeoGebra combines geometric and algebraic features from CAS and DGE. CASs were originally developed in the 1960s for scientific professions and later became commonly used in universities for complex mathematical computations (Roanes-Lozano et al., 2003), thus disseminating into mathematics education. The first DGEs were developed in the 1990s with the didactical purpose (Roanes-Lozano et al., 2003) of teaching and learning Euclidian geometry and later, coordinate geometry and measuring and calculating geometric objects (Oldknow, 2002). Consequently, the syntax of DGE is more accessible for learners than CAS. CASs and DGEs have increasingly incorporated features from one another, and some software has aimed at a fuller integration, such as TI-Nspire and GeoGebra (Freiman, 2014; Sutherland & Rojano, 2014).

The potential of multi-representation is one of the features of DGE and CAS often highlighted in the literature (Drijvers et al., 2010). Representations are instantaneously translated or treated, and a vast number of examples can easily be generated due to dynamic features, such as dragging of objects and sliders controlling variables and animations. Such interaction results in feedback in the form of computation results or translations between representations (Bikner-Ahsbahs et al., 2023; Bokhove & Drijvers, 2012; Olsson, 2018). The feedback allows students to explore, explain, or verify results and conjecture about the underlying rules or patterns, explaining results or translations (Kaput, 1992; Kaput & Schorr, 2007; Moreno-Armella et al., 2008).

In DGE, this involves dynamic constructions of robust figures or measurements of geometric objects for the exploration of relationships within the graphic representation of Euclidean geometry (e.g., Baccaglini-Frank, 2019; Højsted, 2021; Leung, 2008; Leung, 2014; Leung et al., 2013; Mariotti, 2012; Olivero & Robutti, 2007).

In CAS, the translation between representations is more commonly investigated, e.g., students' exploration of functions and their graph (e.g., Artigue, 2002; Bach, 2022; Bloch, 2003; Granberg & Olsson, 2015; Guin & Trouche, 1998; Trouche, 2003). The algebraic features of GeoGebra can also be used to explore translation between registers and conjecture about symbols in an educational tool, rather than a professional tool. Nevertheless, this has mainly been explored with respect to functions (e.g., Bach, 2023; Binti Misrom et al., 2020; Granberg & Olsson, 2015) and rarely with a variable as a general number.

According to my own review (Paper 1), the artifacts of GeoGebra's algebra view, and how they can aid in reasoning in relating to variables as general numbers, are not well understood. Few studies have been conducted on this topic, such as those by Soldano and Arzarello (2017) and Tanguay et al. (2013). They have shown that using a slider tool to assign numeric values to a variable can help students make conjectures about algebraic relationships, such as proportions and ratios. The slider allows students to visually test these conjectures. Moreover, Lagrange and Psycharis (2011) observed students solving LOGO tasks and found that when students can manipulate algebraic expressions, they are more likely to make conjectures about the symbolism in algebraic relationships.

One of the phenomena discussed in relation to students' use of digital tools for reasoning is their reliance on empirical knowledge (Harel & Sowder, 2007) or phenomenological evidence (Baccaglini-Frank, 2019). To counter this tendency, Laborde and Laborde (1995) advocated for designing tasks for DGE that could only be solved using geometrical knowledge. The same argument can be applied to algebra tasks in a DGAE. However, this approach does account for the epistemological gap (Sabena et al., 2014). Teachers may expect a theoretical argument while students approach their answers experimentally, trying to make sense of what they see and explore on the screen. Therefore, another approach is to capitalize on students' reliance on empirical knowledge. For instance, Olive et al. (2010) argues that "by observing properties of invariance simultaneously with manipulation of the object, there is potential to bridge the gap between experimental and theoretical mathematics" (p. 150).

2.5.1 The interface of GeoGebra classic

GeoGebra offers a range of mathematical education tools and resources. The online version of GeoGebra classic was used for this study (see Figure 7). Teachers or researchers can customize the software environment by, e.g., creating restricted environments or designing templates and resources, tailoring the software toward specific student groups and learning objectives.

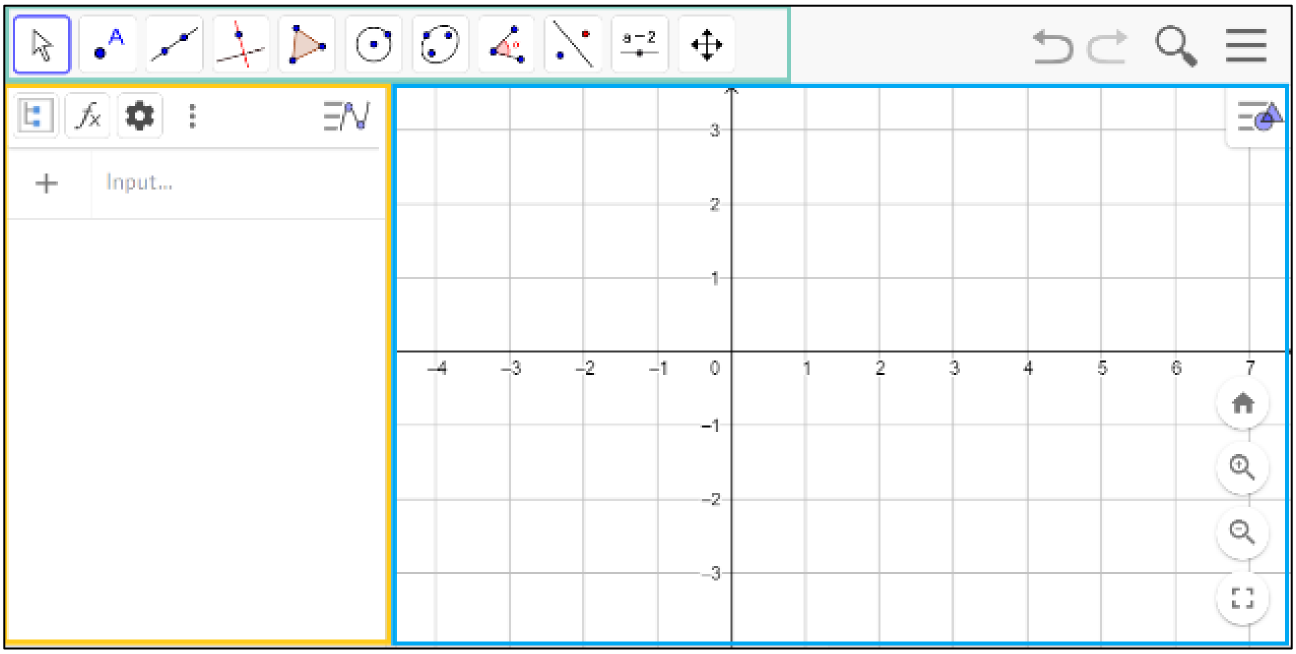


Figure 7 – GeoGebra's classic interface: green frames indicate the toolbar, yellow frames indicate the algebra view, and blue frames indicate the graphic view

The interface is flexible and allows for initiating different views. Views can be enabled and disabled, and various settings can be altered to change the appearance. For instance, the graphic view can be displayed with or without a coordinate system. Additionally, the algebra view can report values, definitions, and descriptions of created objects.

This study utilizes two views: the graphic view (blue frame, Figure 7) and the algebra view (yellow frame, Figure 7). The algebra view has an input bar for commands to create objects, perform computations, and take measurements. A complete list of available commands can be found at https://wiki.geogebra.org. Once a command is input, the output is displayed in the algebra view, while the objects created are represented in the graphic view. The graphic view provides a visual representation of objects, which can be manipulated by dragging them across the screen, depending on the construction constraints that determine their dynamic movement. The toolbar (green frame, Figure 7) contains various menus with categorized tools that enable the interactive construction of objects and measurements in the graphic view.

2.5.2 The representation of variable as a general number in GeoGebra

In GeoGebra, there are two ways in which variables as general numbers appear: implicitly or explicitly (Gregersen, 2022). In this study, the explicit use of a variable is used, since the implicit use can result in representational structures that are algebraically difficult to understand for the age group (Jackiw, 2010; Mackrell, 2011). Explicit variables appear as slider tools that control and assign a value to a variable. By moving a point on the slider, the numerical value changes accordingly. This variable can be referenced in other objects. See and try an example here: https://www.geogebra.org/m/ybywwzha.

Implicit variables can be achieved in two ways. Dynamic objects can be constructed with geometric tools or other objects can be referenced when constructing new objects. As a result, these objects are dependent or co-vary. The variable hence exists by the construction rather than as an independent symbolic representation. See and try an example here: https://www.geogebra.org/classic/urgqu3af.

2.6 THE INSTRUMENTAL APPROACH TO MATHEMATICS EDUCATION

Rabardel developed the Instrumental Approach (IA) during the 1990s and 2000s, drawing inspiration from his PhD supervisor, Vergnaud's (1998b) conceptualization of schemes and the research field of cognitive ergonomics. The main topic of his research was cognition related to the use of instruments. Fundamental principles of the IA are:

- the distinction between artifact and instrument

- the concept of instrumental genesis with its two movements dialectically connected: instrumentalization from the user to the artifact and instrumentation from the artifact to the user
- and the conceptualization of instrumental genesis in terms of the elaboration or appropriation of schemes. (Artigue, 2023, sec. 3.1)

The IA, also known as the Theory of Instrumental Genesis, was adapted for mathematics education by French scholars Defouad (2000) and Trouche (1996). Their doctoral theses initiated the Instrument Approach to Mathematics Education (IAME). Defouad's work emphasized the anthropological theory of didactics (ATD) by considering praxeology in institutions. He focused on students' use of instrumental techniques and material signs in human activities, as well as the discourse used to explain and justify these techniques. Trouche's thesis emphasized the cognitive perspective of Vergnaud and focused on the evolution of schemes of instrumented action in the transition from graphic to symbolic calculators. In this study, I follow Trouche's position, which has been developed and expanded in collaboration with colleagues, including Giun (Guin & Trouche, 1998), Drijvers (Drijvers et al., 2013; Trouche et al., 2013), Monaghan (Monaghan et al., 2016), and Artigue (Artigue & Trouche, 2021). IAME has since disseminated into other theoretical traditions, such as the theory of semiotic mediation (Bartolini & Mariotti, 2008) and activity theory (Bikner-Ahsbahs et al., 2023).

IAME is a developmental theory that conceptualizes how a learner utilizes an artifact in activities associated with specific situations (e.g., using a DGE to construct a robust rectangle or using CAS to solve a differential equation). During the learning process, to effectively utilize the relevant components of the artifact for a given situation or task (e.g., the right angle tool in a DGE or the solve/de-solve functionality of a CAS), these components are transformed into an instrument for the learner (Artigue & Trouche, 2021). The construct of an instrument is distinct from the artifact. An artifact, whether material or non-material, is a product of human creation that carries cultural and social significance (Drijvers et al., 2013). An instrument is a hybrid construct with components of the artifact and cognitive components in terms of schemes (to be elaborated subsequently). The construct *instrument* thus draws on the psychological tradition of considering tools and aids functional extensions of the body and mind (Rabardel & Bourmaud, 2003). Developing an instrument is not a trivial endeavor. Imagine learning to play the trumpet, drive a car, or use a new piece of software. It takes time and effort to understand the mechanics and obtain fluency. This process is called instrumental genesis.

The instrument serves as a mediator in human activities, providing meaning and facilitating the interaction between individuals and their environment (Drijvers et al., 2013). Rabardel and

Bourmaud (2003) assert: “It is not only the artifact that mediates: the instrument is at the heart of mediated activity”. This distinction might seem insignificant, but it emphasizes that the mediated activity has a cognitive component. Mediation can take various forms, including pragmatic and epistemic. Pragmatic mediation happens when the student performs actions directed toward the object, such as measuring the sum of the interior angles of a triangle in a DGE. Epistemic mediation happens when the instrument is used as a means for the object to provide knowledge to the students. For example, if the student measures angles, drags the triangle’s vertices, and realizes that the sum is always 180°. This example shows both the cognitive and artifactual components of mediation. It is made possible by the measuring tool in the DGE and its dynamic features, as well as the student’s knowledge of how to perform the necessary actions and their conceptual understanding of angles.

In Drijvers et al. (2013), instrumental genesis is described as consisting of three dual relationships. The first of these is between the *subject* and the *artifact*, as described above. The second is the relationship between *instrumentation* and *instrumentalization*.

Instrumentation is the influence of a student’s actions on their own knowledge in the process of learning to use an artifact and turning it into an instrument for a task. (Drijvers et al., 2013; Trouche, 2020). Thus, instrumentation is considered “not only as an action (by which someone acquires an instrument) but also as the influence of this action on a subject’s activity and knowledge” (Trouche, 2020, p. 404). In addition, the artifact’s affordances and constraints influence the subject’s activity.

Instrumentalization is less described and researched (Trouche, 2020). It is directed from the subject toward the tool, as the student influences or even disrupt the artifact in the process of instrumental genesis. “Instrumentalization can thus lead to enrichment of an artifact, or to its impoverishment” (Trouche, 2005, p. 148).

These two dualities are consistent with how Trouche has previously described instrumental genesis. However, the third duality *scheme-technique*, has been more controversial. I will discuss the controversy in the next section. This relationship particularly highlights Vergnaud’s influence on IAME. Following Vergnaud (1997), *schemes* are defined as “*the invariant organization of behavior for a certain class of situations*” (p. 12). As the subject learns to use specific components of the artifact for classes of tasks or situations, specific techniques and underlying schemes that control these techniques take shape, expand, and solidify. The thinking process is perceived as perceptual and gestural activity that unfolds over time and adheres to a particular structure, but activity itself *is not invariant*. Indeed, even though activity involves rules governing outward behavior and internal cognitive processes, rules are adapted based on the specific context. Schemes consists of different elements:

- Goals, subgoals, and expectations
- Rules of action: They can be considered the generative component of schemes, responsible for generating behavior based on situational variables.
- Operational invariants: They primarily involve concepts-in-action (to categorize and select information) and theorems-in-action (to infer appropriate goals and rules from the available and relevant information).
- Possibilities of inference: These possibilities are essential since inference and computation are inherent in any activity (Vergnaud, 1998b, p. 229).

Drijvers et al. (2013) then consider the *technique-scheme* duality to be the relationship between gesturing and thinking. However, the use of the term *technique* has been debated, and I will contribute to this debate with my own position in the next section.

2.6.1 Techniques in the instrumental act

As the early development of IAME was influenced by French traditions, the term *technique* was adopted from ATD but used and developed in relation to *scheme*. However, the adaptation of *technique*, as described by Artigue (2023), has been heavily debated. The essence of the debate concerns the theoretical reduction and the incompatibility between their *background theories* (Mason & Waywood, 1996). The ATD analyzes cognition in mathematics as praxeology, which consists of the practical set: a type of task and a technique (to solve such tasks), and of the logos set: a technology (the terminology) and a theory (the reasoning). Together they are represented by the quadruplet $[T, \tau, \theta, \Theta]$ (Chevallard, 2019). Consequently, in ATD you cannot talk about a technique without the context of the other three elements. As elaborated above, IAME considers cognition in terms of schemes. Remember that a scheme is defined as specific to types of situations or types of tasks. This part is consistent with ATD as the practical set. However, instead of the logos set, the IAME operates with schemes. This is the quarrel.

In IAME, techniques are explained with reference to the ATD (e.g., Trouche, 2005) but also as a particular organized set of gestures "distinguishing an elementary level of command constraints and a more complex level of organization constraints [...] within students' activity, between a level of gesture and a level of technique" (Trouche, 2005, p. 147). Trouche (2005) relates techniques to schemes and considers an instrumented technique to be a technique that integrates one or several artifacts, which are guide and form an *instrumented action scheme*.

However, Artigue (2023) critiques this:

> ... thinking in terms of praxeologies means that techniques cannot be isolated from the technological discourse that describes, explains and justifies them. In a sense,

> reducing techniques to gestures is akin to reducing schemes to their observable characteristics without considering the essential component of schemes that the operational invariants underlying the observed regularities are. Indeed, the many contributions to the scheme/technique debate have made it clear that schemes and techniques correspond to two different and complementary ways of approaching instrumental issues, both insightful but irreducible to each other. (p.33)

Still, Drijvers et al. (2013) describe techniques as gestures and consider schemes and techniques parts of instrumental genesis. Though acknowledging the theoretical reduction of adopting the term *techniques*, they take a practical stance:

> we see techniques as the observable part of the students' work on solving a given type of tasks (i.e., a set of organized gestures) and schemes as the cognitive foundations of these techniques that are not directly observable, but can be inferred from the regularities and patterns in students' activities. (Drijvers et al., 2013, p. 27)

They also elaborate on techniques as carriers of both practical and theoretical knowledge, as is the premise of ATD. In doing so, they acknowledge the analytical need for a construct that connects students' gestures, particularly those related to artifacts, to students' cognition. This approach disregards theoretical reductionism and emphasizes the need to conceptualize gestural activity performed on the artifact within the instrumental act, which involves the hybrid entity *instrument* (artifact + scheme). This distinction is important for separating activity that only involves schemes from activity that incorporates the instrumental aspect.

In papers 3, 4 and 5, I use the term *technique* with reference to the scheme/technique duality (Drijvers et al., 2013), which is an expression of the same "practical" need. To address the theoretical reductionism, I do, however, find it necessary to reinterpret the term *technique* anchored in the background theory of IAME, parting with IAME's link to ATD. This requires considering the original work on the Instrumental Approach by Rabardel and colleagues (e.g, Rabardel, 1995/2002; Rabardel, 2001; Rabardel & Bourmaud, 2003) and approaching *techniques* through Vergnaud's definition of scheme.

In the following, I explore the terms used to describe gestural activity by Rabardel and colleagues and relate these to the notion of scheme as defined by Vergnaud (1998b). Rabardel (1995/2002), and later Rabardel and Bourmaud (2003), describe the organization of activity in the mobilization and implementation of schemes as *usage modalities* or *activity modalities* of an artifact. When they operationalize the notions in analysis, modalities are presented as steps (organization) in a series of specific activities related to an artifact. Each step describes the goal and decisions of the subject, and the objects of the activity. Changes in the usage modalities indicate the development of schemes. Therefore, techniques are at least *usage modalities* of artifacts that require the mobilization and implementation of an *instrument* (following the definition of instrumental genesis). However, as Rabardel expresses it: "it is necessary to analyze and understand what these activities are from the perspective of the users themselves" (1995, p. 31). Indeed, from an RC perspective, understanding the student's perspective in the development of schemes is of prominent concern. A more inclusive approach would be to consider techniques as encompassing all gestures involved in student activity when using a digital tool (such as hand movements during expressions, observing an artifact or activity on objects mediated by the artifact, or articulation of imagined activity). This allows for a richer description of students' cognition than focusing solely on *usage modalities*.

Rabardel (1995) and Rabardel and Bourmaud (2003) also distinguish between two types of *utilization scheme*s and their related activities: usage schemes expressed through secondary activities aimed at managing the artifact (such as selecting specific colors or changing the number of displayed decimals), and *instrumented action schemes*, which involve primary activities oriented toward the execution of specific tasks. As the primary activities are those that involve mathematical concepts, I find it beneficial to limit techniques to primary activities.

In summary, I consider techniques the primary perceptual and gestural activities that involve the mobilization and implementation of instrumented action schemes. In summary, I consider techniques as the primary perceptual and gestural activities that involve the mobilization and implementation of instrumented action schemes. This is illustrated in Figure 8.

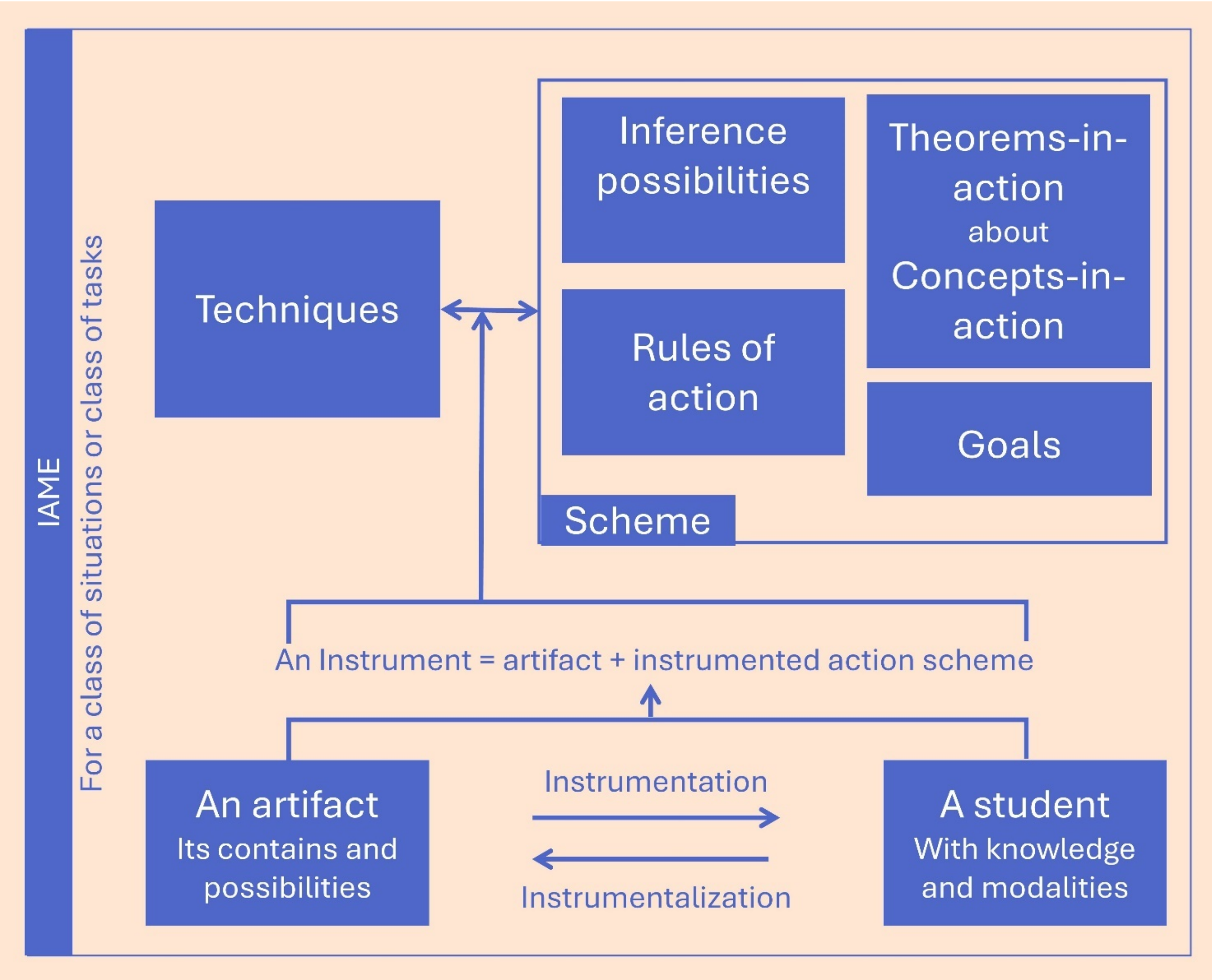


Figure 8 - Illustration of instrumental genesis and instrumented action scheme placing techniques as a primary activity

2.6.2 Artifacts and tools

In the IAME literature, a specific piece of software or device is most often referred to as the artifact, e.g., a CAS with functionalities. Using different functionalities then means using part of the artifact (Trouche, 2005). The notion *tool* applies when the artifact is used (Monaghan & Trouche, 2016). In other instances, tools and functionalities in the software are described as artifacts, forming part of a collection of artifacts (e.g., Drijvers et al., 2013; Leung, 2008a).

> What exactly is the artefact in a given situation is not always clear: for example, in the case of dynamic geometry software, it is a matter of granularity if one considers

> the software as one single artefact, or if one sees it as a collection of artefacts (Drijvers et al., 2013, p. 26).

Therefore, clarifying how I use artifacts and tools is necessary. While I do consider GeoGebra an artifact, it is one that structures different interrelated views (see Figure 7 and Figure 9), such as the graphic view and algebra view. Each view embodies and organizes different domains or subdomains of mathematics, each with its own syntax. I will refer to GeoGebra as a software organized in different views with a collection of artifacts. Artifacts become tools when used, and they become elements of an instrument through the process of instrumental genesis. Artifacts in DGAE create representations of mathematical objects, and one could argue that even the representations are tools for solving types

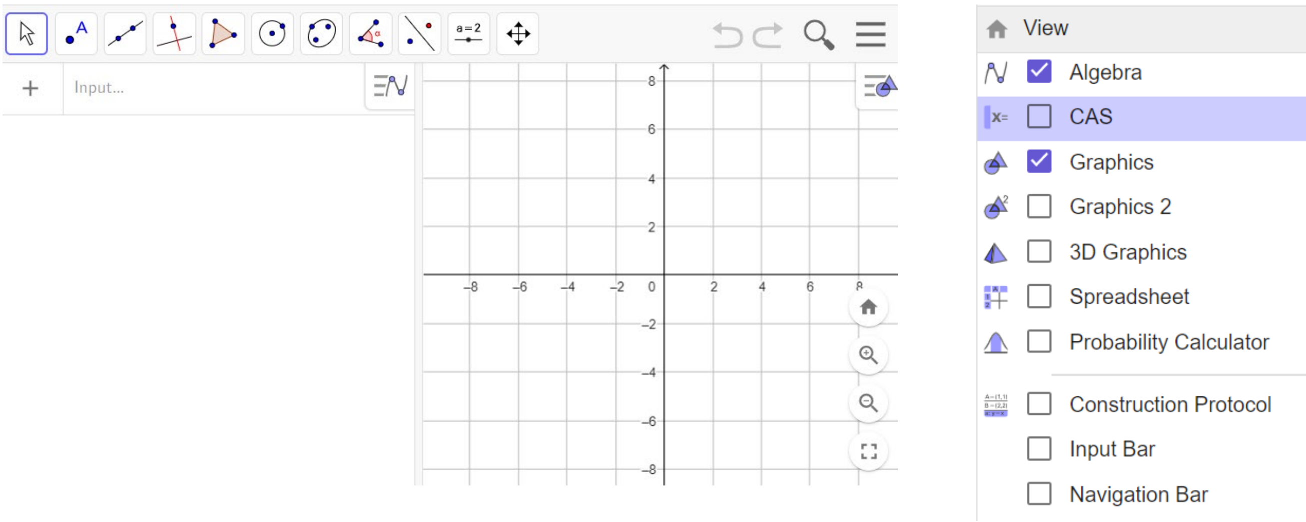


Figure 9 – Left: The algebra and graphic view. Right: The menu option showing the views of GeoGebra

of tasks. However, in this context, I will not consider representations artifacts. An artifact can be described in terms of its functions. As an example of using the construct, the algebra view is used by typing commands into its input field. I consider each of such commands an artifact. For example, there is a command for creating a point, a function, a geometric object, yielding the divisors of a value, and so on; each is an artifact. The slider artifact that creates and controls the value of a variable comes along with functions such as dragging a point on the slider to change the variable's value and the animation function, which automates changing the value and setting the limits of the variable.

2.6.3 Classes of situations and a specific type of tasks

Schemes are only relevant for a specific class of situations or specific types of tasks (Trouche, 2005). In this study, a type of task can be considered from the perspective of the two nodes in the pentahedron (see section 2.2): the mathematical activity, in this case justification, or the mathematical content, concerning variables in ordered pairs. I return to this in Chapter 6. The latter type is more typical in the literature. For example, in Trouche (2004) the type of task is to solve an equation with two unknowns.

This chapter have described and discussed perspectives on theory and elaborated on the theorical foundation of the study. Drawing on this, the following chapter present the research questions of the study.

3 RESEARCH QUESTIONS

The study's theoretical and practical aim resulted in the formulation of three research questions (presented below) based on the previously described and discussed theoretical perspectives. Each research question is underpinned by specific assumptions or hypotheses, which I will present and argue for here.

Because the project concerns the use of DGAE in relation to students' exercise of mathematical RC, it is relevant to create situations for students to exercise their RC and learn about something mathematical. The mathematical topic in focus is variable as a generalized number in basic algebraic expressions along with basic algebraic properties such as equality, infinity, limits, and structural relationships (e.g., multiplicative and additive) that can be expressed when a variable is used in simple algebraic expressions.

GeoGebra contains artifacts, such as the slider tool, that allow operations on variable expressions to be converted into changes in graphic representations that appear as (virtual) real objects. My "naïve" hypothesis is that GeoGebra's dynamic multi-representation of algebraic, numeric, and graphic representation has the potential to support and encourage students to exercise their RC regarding algebraic properties and relationships by relating those representations in justification processes. However, for students to capitalize on those affordances, tasks must support such justification processes in their interaction with GeoGebra. To explore how such tasks can be designed, the first research question (RQ1) inquires:

> In what ways can tasks be designed to encourage lower secondary students to exercise their reasoning competency when using a dynamic geometry and algebra environments in the case of justification focusing on variables as a general number?

To gain insight into students' justification, in the context described above, as they work with the tasks developed for RQ1, the second research question (RQ2) explores:

> What are the relationships between lower secondary students' scheme-technique duality when solving tasks developed for RQ1 in a dynamic geometry and algebra environment and their exercise of reasoning competency as justification?

By RQ2, I assume that differences in students' scheme-technique duality when solving the given tasks can be related to differences in students' exercise of RC. Certainly, students exercise their RC with different complexity, which is assessed by the three dimensions: the degree of coverage, the radius of action, and the technical level(Niss & Højgaard, 2011, 2019). However, such an assessment of students' RC alone does not shed light on their engagement with the DGAE in the justification processes. Likewise, the IAME does not have notions that particularly relate to reasoning; how these two theoretical perspectives can be linked to capture relationships will require theoretical consideration and development. In RQ2, I hypothesize that IAME and RC are related. Such analysis might serve as the opportunity to uncover links, where each theoretical approach has methods to capture different aspects of students' tools used in justification processes. (Niss & Jankvist, 2022). Hence the third research questions (RQ3) address:

> Which theoretical links can be established between reasoning competency and the Instrumental approach to mathematics education from the theoretical developments of the study?

The three research questions are interconnected and mutually influential, suggesting that outcomes that pertain to one question may also be relevant to the others. Given the need to develop tasks tailored to specific scenarios and to make substantive contributions to the research field both empirically and theoretically, design research serves as the overarching methodological framework for the project. To enhance the theoretical underpinnings, the study incorporates a networking perspective into the theoretical discussions and reflections.

The next chapter accounts for design research and role of theory.

4 Educational design research

This chapter elaborate on the methodology of the project, which is educational design research (DR). 4.1 describe DR and its characteristics. 4.1.1 comment on theory development in DR, and finally, 4.1.2 discuss the quality of empirical research and DR.

4.1 Educational design research

DR is referred to by different names, including 'design studies', 'didactical design research', 'design experiments', 'design-based research', 'design research', and 'design engineering'. DR combines instructional design, which develops sequences for teaching and learning in an educational setting, and educational research, which addresses teaching and learning processes aiming to understand these processes and develop theory. Instructional design and educational research are combined into an iterative cyclic process toward maturing both design and theory (Bakker, 2018; Cobb et al., 2003; Gravemeijer & Prediger, 2019; McKenney & Reeves, 2014). It is the dual aim of DR that makes this methodology particularly relevant for this study, as it allows for answering both RQ1 and RQ2 and provides a foundation for RQ3.

Each notion implies differences in methods, but core characteristics are:

- aiming to develop both theory and practice through a design within realistic settings
- designing for the development of educational theory
- being interventionistic in nature, having both prospective and reflective components
- being cyclic, emerging from iterative conjecturing, testing, and revising
- developing theory that is transferrable to other contexts
- being pragmatically rooted (Bakker, 2018; Barab & Squire, 2004; Cobb et al., 2003; Gravemeijer & Prediger, 2019)

However, DR can vary in many ways, e.g., in the applicability of the practical results, the scope of the theoretical results, the intensity of collaboration with practitioners, the number of iterations, and the theoretical anchoring and theoretical approaches in the study (Bakker, 2018; Gravemeijer & Prediger, 2019). Particularly, the aim of a given study influences the emphasis on either theory development or practical implications. Studies with a practical emphasis are primarily conducted to:

> "• Solve a problem (e.g., increase the participation of women and other minorities in engineering and science careers),
>
> • Put knowledge to innovative use (e.g., use the affordances of smartphones to enable mobile learning), and/or

• Increase robustness and systematic nature of design practices (e.g., establish a set of design principles for implementing inquiry-based learning in middle school science).” (McKenney & Reeves, 2014, p. 133)

Studies with a theoretical emphasis are primarily conducted to:

“• Generate new knowledge (e.g., develop a theory of game-based learning),

• Generate different types of knowledge (e.g., enhance and extend knowledge related to professional development for scaffolding strategies for math teachers), and/or

• Increase the ecological validity of research-based knowledge (e.g., increase the likelihood that educational innovations will be used to transform educational practice).“ (McKenney & Reeves, 2014)

Topic-specific studies (Gravemeijer & Prediger, 2019) will often investigate pathways through a given topic, which often produce hypothetical learning trajectories (Bakker, 2018). Some studies are explorative, as little is known, and study the phenomena and opportunities that arise in the testing of the design.

The present PhD study has a theoretical emphasis because the overarching aim is to develop theoretical knowledge of students' exercise of RC as they use digital environments which is addressed by RQ2 and RQ3. The practical results are expressed in the answer to RQ1 by “putting knowledge to innovative use”, as the study focus on students' use of the algebra view but also “increase robustness and systematic nature of design practices” in the development of some design principles. The cycle of DR studies has specific phases that make up one iteration toward maturing the design and toward developing coherency and detail in the theoretical elements and their implications for the theoretical approaches in the fields. As illustrated in Figure 10, the phases are

1) analysis and exploration toward constructing and designing
2) testing the design to gather data in classrooms

3) retrospective analyses by evaluation and reflection

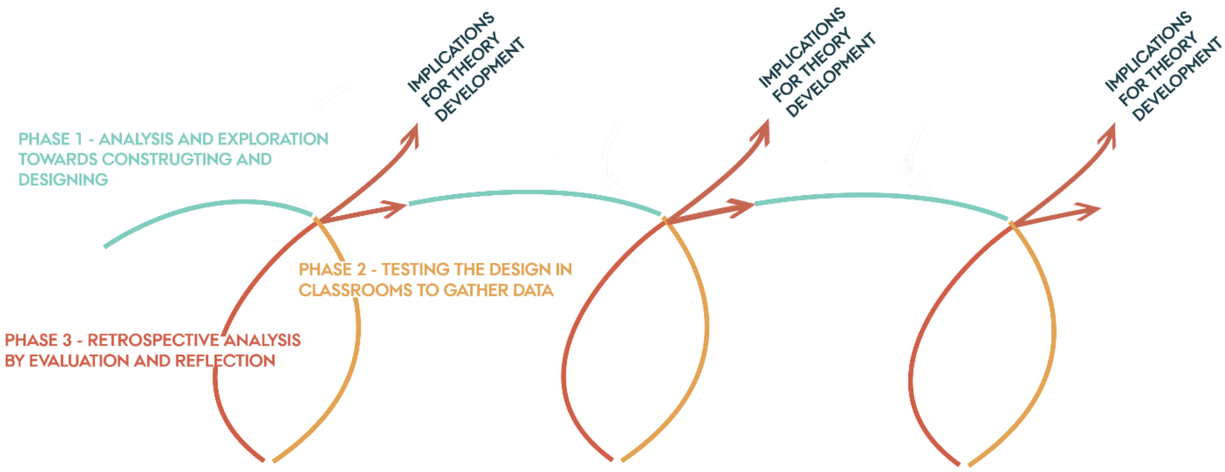


Figure 10 – Iterative phases of design research

Different authors "cut" or cluster these phases differently, as each phase relates to processes in the other phases (e.g. Bakker, 2018; diSessa & Cobb, 2004; McKenney & Reeves, 2018). The phases are cyclic, but each phase can be "revisited" in micro-cycles within a single iteration

McKenney and Reeves (2018) emphasize how the researcher should take both analytic (detective) and creative (inventor) perspectives in all phases: "The detective is highly rational, seeking knowledge that is grounded in evidence from the field and supported by scientific understanding … By contrast, the inventor is original, striving to innovate and embracing opportunity" (p. 89).

Initially, phase 1 entails activities such as identifying, formulating, and exploring an educational problem that can best be answered through DR. The analyst seeks to define and understand the problem by reviewing literature and receiving feedback from collaborates, and the inventor seeks inspiration and ideas that uncover opportunities toward a solution of the problem (McKenney & Reeves, 2018). In this phase, a preliminary or hypothetical theoretical lens is developed (Gravemeijer & Prediger, 2019), guiding the emerging of an early design and informing the formulation of hypothetical and humble design principles (see subsection 4.1.1). In the following iterations, these elements will be refined and adjusted according to phase 3. Still, the creative perspective allows for creativity in their application and serendipity (McKenney & Reeves, 2018). The design can concern materials such as computer tools, tasks, activities, or learning environments but also principles of how students or teachers are expected to act or communicate to obtain set goals (Bakker, 2018). Designing encompasses the exploration and mapping of solutions, and construction encompasses the creation and development of prototypes (McKenney & Reeves, 2018). Phase 1 should consider students' prior knowledge and appropriate learning goals, as well as the teaching-learning strategy (design) that can assist students toward this goal (Bakker, 2018).

The purpose of phase 2 is to obtain data that provide information to improve the envisioned design, allowing for testing and revising conjectures (Cobb et al., 2009). The preliminary lens frames the inquiry so that appropriate data is collected and follows appropriate methods. The researcher is often deeply involved in the execution of the intervention in collaboration with the practitioners. This both presents great opportunities to discover challenges and potentials of the design that can spur critical reflection, but also a methodological issue as the involvement influences the results (McKenney & Reeves, 2018).

In phase 3, the data are analyzed, and the design is evaluated toward refining theoretical assumptions and design principles (Cobb et al., 2003; Prediger, 2019). It may involve exploring phenomena that the intervention is known to engender (McKenney & Reeves, 2018). Collins et al. (2004) argue, "it is important to identify the critical elements of the design and how they fit together. In order to evaluate any implementation, one needs to analyze each particular case in terms of these key elements and their interactions" (p. 34). This may involve considering the success of both the implementation and its results (Bakker, 2018). Ejersbo et al. (2008) remark that ideally, the development of design and the development of theory run simultaneously, but that this ideal can be difficult to practice. Typically conjectures evolve during and after teaching experiments, leading to loops between designing and conjecturing (Confrey & Lachance, 2000).

4.1.1 Theory in design research

Different types of theory elements have different functions in DR, and some are developed in turn during a DR study.

Categorial theory elements provide a language with logically structured descriptive concepts and are used to understand and distinguish phenomena and relations (Prediger, 2019). They have functions similar to a background theory (subsection 2.1.2) as it "is decisive for all further theory elements, as they provide the language to describe, set aims, and explain or predict in propositional theory elements" (Prediger, 2019, p. 9). In the present study, the theoretical framework described in Chapter 3, the KOM framework along with the IAME, has elements that provide such language, e.g., that mathematical mastery is understood in terms of mathematical competence, constituting a set of logically structured descriptive concepts, and that tool use is a genesis of human and artifact.

This also reflects in the *normative theory* elements, which state and justify aims and principles such as learning goals or process qualities within a given context and elaborate on their foundations (Prediger, 2019). In this PhD project, the normative element provided by the KOM framework is anchored in the proposition that mathematical mastery is understood as mathematical competency (Niss & Højgaard, 2019; Niss & Jankvist, 2022) and expressed in the goal of students to exercise their RC. Other normative elements have also been included to provide an informed foundation for the design and construction of tasks. These elements are explained in Chapter 6.

Descriptive theory elements serve to describe the quality and occurrence of certain phenomena and relationships. These can be features, hierarchies and frequencies of different categories. Identifying and refining descriptions of phenomena is a typical step in DR (Prediger, 2019).

Explanatory theory elements explain certain phenomena by pointing to cause and effect between phenomena or structures. Categorical and descriptive components are needed to develop explanatory elements, though in empirical research they are often co-developed (Prediger, 2019). If a relationship between descriptive elements can be derived, it increases the explanatory power.

Predictive theory elements justify certain solutions/actions toward a given aim or problem, or they predict outcomes of action, design elements, or structural elements. In DR, predictive theory elements are traditionally developed as design principles. Van den Akker (1999) propose that design principles follow this structure:

> If you want to design intervention X [for the purpose/function Y in context Z], then you are best advised to give that intervention the characteristics A, B, and C [...], and to do that via procedures K, L, and M [...], because of arguments P, Q, and R. (p. 9)

Akker's formulation combines the *how* and *why* underpinning the dual aim of DR. The arguments in support of the design characteristics and procedures can both be empirical and theoretical (Bakker, 2018).

To emphasize the hypothetical nature of early design principles, I label them humble design heuristics (HDH), following Prediger (2019). The developing theoretical lens guides the early HDH hypotheses, which are developed into principles as they mature in the retrospective analysis of phase 3. The design principles are continuously revisited throughout the kappa, to explicate how different processes have provided new insights to progress the principles.

4.1.2 Quality in research and design research

Schoenfeld (2007) has established quality criteria for validity and reliability within empirical research, which also extend to DR (e.g., Højsted, 2021; Jankvist, 2009). I will discuss these criteria regarding the project in the final discussion. Schoenfeld (2007) argues that research must be examined based on three dimensions: credibility, generalizability, and importance. Ensuring quality in DR involves such key aspects, including trustworthiness, descriptive and explanatory power, and the generalizability and transferability of results. This section explores these dimensions to provide a comprehensive understanding of quality in DR.

In DR, trustworthiness includes both validity and reliability. Bakker and van Eerde (2015) underscore the significance of internal and external validity in establishing the credibility of research findings. Internal validity aligns with Schoenfeld (2007) descriptions and pertains to the quality of the data and the soundness of the reasoning leading to conclusions. To fortify internal validity, researchers often employ data triangulation in retrospective analysis, incorporating diverse data sources such as transcripts, videos, screencasts, and written products. This multiplicity of data sources allows for a robust examination of the research findings.

Reliability in DR can be challenging due to the complexity of naturalistic settings where interventions are implemented (Collins et al., 2004). The numerous dependent and independent variables in such environments can affect the consistency of results. Despite these challenges, maintaining rigorous methodological standards and ensuring transparency in data collection and analysis processes can bolster reliability. Schoenfeld (2007) notes that findings with less internal validity may only serve as proof of existence, underscoring the need for stringent validity checks in DR studies.

Documentation and description of the DR processes ensure rigor, specificity, and replicability. Schoenfeld (2007) adds that the description should concern only essential aspects. Descriptive and explanatory power in DR refers to how well the research can describe and explain phenomena within the study context (Prediger, 2019). Predictive power in DR involves the ability to forecast outcomes based on the design principles developed during the research. These principles, while predictive, require rigorous testing and validation to ensure their applicability across different contexts. Schoenfeld (2007) asserts that theoretical claims need to be testable and refutable, enhancing their potential for refinement and validation.

Generalizability in DR pertains to the extent to which research findings can be applied beyond the original study context. Schoenfeld (2007) outlines four types of generalizability: claimed, implied, potential, and warranted. Claimed generalizability is the set of conditions explicitly stated by the researchers as applicable, while implied generalizability is suggested indirectly. Potential generalizability refers to contexts where the findings might reasonably apply and warranted generalizability is substantiated by trustworthy evidence. In the context of DR, generality is thought of as the transferability of results (Bakker, 2018). These theory elements include categorical, normative, descriptive, explanatory, and predictive elements, each serving different functions within the research framework (Prediger, 2019). Schoenfeld (2007) emphasizes the importance of the relevance and significance of research results for both theory and practice. The importance of the results, therefore, lies in their ability to advance the field and inform future research and practice in mathematics education.

5 OVERVIEW OF THE EXECUTION OF THE STUDY

This chapter provides context for answering the research questions (1–3). 5.1. present an overview of the phases of the study, and position the study in the landscape of DR. 5.2 account for the data collection. 5.3 summarize the papers included in the kappa.

5.1 EXECUTION OF THE STUDY

The project has followed the iterative structure of DR (see chapter 4), emphasizing linking KOM with IAME. As Figure 11 illustrates, the retrospective analysis of the DR serves as a foundation for the networking of the two frameworks KOM and IAME, which again influence the redesign and construction of tasks. The development of the task design is described concretely and thoroughly in Chapter 6.

Iteration 1: Based on an initial literature review, the preparation phase explored possibilities for students to use artifacts of the algebra view for justification through classroom experiments of seven explorative tasks. Networking efforts explored mediation processes as expressions of RC but pointed to “missing links” between KOM and IAME.	Iteration 2: The design and preparation phase led to the emergence of a “microworld” designed around the idea of “variable points” and the construction of a sequence of tasks. The retrospective analysis identified tasks with possibilities for students exercise of. Networking efforts to link KOM and IAME produced an analytical model of student justification processes using GeoGebra.	Iteration 3: The task sequence was redesigned and tested, leading to potential developments in task design. Common difficulties among students exercising RC and using algebra were revealed. Networking efforts deepened connections to Vergnaud’s scheme concepts and considered three dimensions of RC relations..

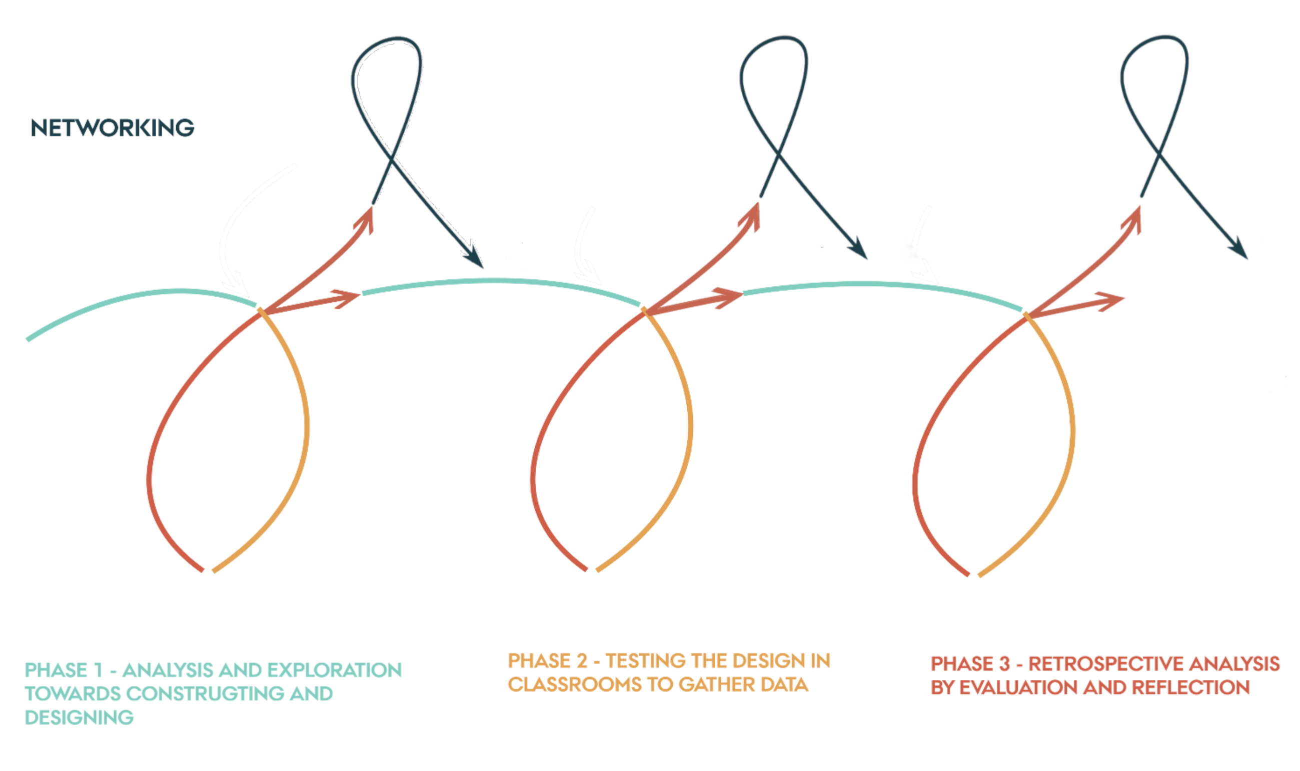


Figure 11 – Illustration of the research process as an interaction between DR and NT

5.1.1 Characterizing the study within design research

The processes of a DR study are highly influenced by the aim of the study and its emphasis on the practical or theoretical perspective. As already indicated, this project emphasizes the theoretical perspective. This does not mean that the practical perspective is devaluated; Indeed, the theoretical developments obtained in the study rely on the quality of the empirical evidence.

The theoretical aim of this study is to enhance and extend the knowledge of students' RC and use of digital tools. This aim also emphasizes why networking of theories are relevant. As the KOM framework is a medium-level framework (this is argued in section 6.5), it does not capture the fine-grained processes of interplay between DGAE and students' reasoning processes. To gain insight into this interplay, the theory development must therefore consider the KOM framework and draw on the theories of MER with methods and concepts to describe the fine-grained processes. The theoretical development is hence anchored in a practical perspective.

Often a DR study is topic-specific (Gravemeijer & Prediger, 2019). However, it is not the case for this study, as it does not aim to teach students specific mathematical content by creating optimal pathways, such as learning trajectories. Rather the aim is to achieve insight for theory development in cases where students exercise their RC while they use a DGAE. For students to exercise any competency, they must be given a mathematical situation that concern *something* mathematical. Moreover, this must be chosen with particular care toward the target group. Consequently, the practical perspective is to encourage students to exercise their RC with artifacts in GeoGebra's algebra view by considering how to capitalize on the affordances of GeoGebra's algebra view. In addition, to increase the robustness and systematic nature of the design in terms of design principles (Van den Akker, 1999) to be able to develop different types of knowledge, e.g., descriptive, explanatory, and predictive theory elements (Prediger, 2019).

5.2 CLASSROOM EXPERIMENTS AND DATA COLLECTION

Table 1 shows an overview of classroom experiments and the data collection. The two participating schools A and B are public schools. School A is located in a suburb of Copenhagen, and school B is in a suburb of Aarhus. The classroom experiments were primarily conducted in the 7th grade. In the pilot of iteration 1, two PhD fellows also participated. Note also that iteration 3 diverges in its organization due to covid-related issues and includes an 8th grade. I will account for those in this section too.

The classroom experiments were conducted in collaboration with the mathematics teachers of the participating classes, who also assisted with obtaining consent for student participation and informing parents. Before the classroom experiments, I met with the mathematics teacher of the given class to organize the sessions and adjust the tasks. This meeting also prepared both the teacher and myself for guiding and supporting the students, e.g., discussing how to give hints without providing answers and how to support students in clearly accounting for their thinking and justifications. After each classroom experiment session, I also met with the teachers to evaluate and, if necessary, make minor adjustments before the following test. These were primarily specific formulations that diverted from the discourse of the given mathematics classroom.

In the experiment, the students worked together in pairs with one laptop or Chromebook to make students voice their thoughts and arguments and to experience a need to justify their solutions or solutions strategies to a peer. The student's work was documented with video and audio recording software using OBS studio on laptops and WeVideo on Chromebooks, capturing the computer screen, voices, and the students' upper bodies. In addition, in collaboration with the teachers, two focus pairs from each class were selected to be recorded by a stationary camera capturing the computer screen and their hand gestures in front of it.

The students accessed the GeoGebra worksheets for given tasks through GeoGebra groups (now being replaced by GeoGebra classroom). This allowed me to access the final state of the student's work in the GeoGebra group. In class a, the tasks were also posed along with an input field for written answers. In the remainder experiments the tasks were given in Microsoft Word documents for the and only the GeoGebra worksheets were accessed through GeoGebra groups.

Table 1 – Overview of data collection

	Grade	When	Where	Duration	Classes and (students)	Teachers
Iteration 1	7^{th} grade	August 2019	School A	2 lessons of 45 minutes	Class c (23)	Teacher 2
	PhD students	September 2019	Aarhus University	1 hour	(2)	none
Iteration 2	7^{th} grade	October/ November 2019	School A	2 x 2 lessons of 45 minutes	Class a, (21/5) Class b (17) Class c (23)	Teacher 1 Teacher 1 Teacher 2
Iteration 3	7^{th} grade	June 2021	School B	2 lessons of 45 minutes	Class x (4)	None
				2 x 2 lessons of 45 minutes	Class y (25)	Teacher 3 + Substitute
				2 lessons of 45 minutes	Class x (4)	none
	8^{th} grade	June 2021	School A	2 lessons of 45 minutes	Class c (23)	Teacher 2

In iteration 1, seven tasks were tested in one classroom, which also tested the screencast software and the setup of video cameras. Some tasks showing promise of further development in the classroom experiments were also tested on two PhD peers to get a reference in the form of expert answers to the tasks. I took on the role of an interviewer as they solved the tasks.

In iteration 2, not all students agreed to be video recorded. This was particularly an issue in class A, where only five students agreed to be recorded. This resulted in two focus groups and one group that did not record their face and upper body, but only screen and voices. However, all students in the class worked with the tasks, and the teachers and I discussed our impressions of watching and talking with the students, as they solved the tasks, in our evaluation of the sessions. Practically, this means that results from class A were used as a pilot.

Iteration 3 was, to some extent, influenced by covid restrictions. Once the students were fully back in school and schools were open to visitors, the summer holiday was approaching. The experiments had to be adjusted to what was possible before the holiday and the restrictions still enforced. School B had two 7^{th} grade classes, Y and X. Two focus pairs from class X were used as a pilot to decide on particular developments of the task and test the recording software WeVideo. Class Y was to try the full task sequence. The mathematics teacher of class Y was put in isolation throughout the period of the classroom experiments due to a positive covid test, and a substitute participated in class. Consequently, the students had less support with technical issues related to recording, the use of GeoGebra, and task completion. This affected the quality of the collected data, so I organized a focus group session with two pairs of students from class X. I also engaged class C from school A (now in 8th grade) to solve the developed task sequence. Their mathematics teacher conducted this experiment after careful instruction from me.

5.2.1 Publications related to the study

The writing of review papers has been ongoing since in the early stages of the study and has continually developed and shaped the study throughout all three iterations. Besides the papers presented in the kappa, other conference papers and publications have been produced; some in collaboration with colleagues. Table 3 presents an overview of how these productions fall within the three iterations.

Table 2 - Overview of publications associated with each cycle

	Focus	Publication
Review studies	Providing groundwork for existing knowledge on the use of tools in the algebra window of GeoGebra for reasoning processes, networking of theories, and representation competency. The reviews have been initiated in iteration 1 but have been ongoing until publication.	Pedersen et al. (2021) **Paper 6:** Bach et al. (2021) **Paper 1:** Gregersen (2022)
Iteration 1	Exploring mediation processes	Gregersen (2020)
Iteration 2	Exploring IAME toward capturing justification processes and other potential frameworks for reasoning. This leads to the development of an analytical model to capture justification processes in students' use of a DGEA, also exploring the potentials of the algebra view and a definition of instrumented justification.	**Paper 2:** Gregersen and Baccaglini-Frank (2020) **Paper 3:** Gregersen and Baccaglini-Frank (2022)
Iteration 3	Evaluation of the developments implemented in the tasks sequence by comparing students' exercise of RC and evaluation of specific types of tasks and their potentials related to using the algebra view. Also, liking IAME to KOM through students' use of techniques in their instrumented justification	**Paper 4:** Gregersen (In review) **Paper 5:** Gregersen (2024)

5.3 SYNTHESIS AND RESULTS FROM PAPERS IN THE KAPPA

In this section, I summarize the papers of the kappa and describe how they relate to and build on one another.

5.3.1 Paper 1 - How about that algebra view in GeoGebra? A review on how task design may support algebraic reasoning in lower secondary school

This review paper investigates the potential of GeoGebra's algebra view for task design concerning lower secondary school students' RC when working with variables as a generalized number. Despite an extensive search in the existing literature, only five conference papers on this topic were found, indicating a lack of research in this area. The research on the use of DGE in mathematics teaching and learning has mostly focused on Euclidean geometry and simple and complex functions.

The paper discusses three empirical studies and two theoretical papers that explore either the explicit or implicit use of variables. The results suggest that GeoGebra's integration of geometry and algebra can pose didactical issues. The analytical algebra in the algebra view lies beyond the scope of lower secondary school mathematics (ages 13–15), and the construction of geometrical objects in the graphic view, which produces implicit variables, can result in discrepancies in the representations in GeoGebra (Jackiw, 2010; Mackrell, 2011).

However, with the explicit use of variables (Soldano & Arzarello, 2017; Tanguay et al., 2013) and by providing students with the possibility to transform algebraic expressions (Lagrange & Psycharis, 2011), it is possible to direct students' reasoning toward algebraic expressions. In this context, the use of sliders can validate or refute conjectures about the relations between numeric values and geometric relationships (Soldano & Arzarello, 2017; Tanguay et al., 2013). The slider tool provides a link between graphic representations, algebraic representations, and numeric values, allowing for a more comprehensive understanding of mathematical concepts (Mackrell, 2011).

The paper concludes that, while there is little research on functionalities in GeoGebra's algebra view for working with variables as a general number, using sliders for explicit variables can activate lower secondary students' mathematical RC. Further exploration of typing expressions with variables in the context of GeoGebra is also recommended.

5.3.2 Collective introduction of papers 2, 3, 4 and 5

Together, papers 2, 3, and 5 demonstrate the evolution of theoretical developments aimed at bridging the gap between the KOM framework and IAME to capture students' justification processes when utilizing tools, while paper 4 focuses on task design for RC. They all rest on the results of the review, and with different perspectives, they add to our understanding of students' use of artifacts in the algebra view in the context of reasoning about variables as a generalized number and properties of the variable.

In paper 2, a conference paper, the initial attempt to connect the two frameworks is introduced by interpreting the scheme-technique duality (Drijvers et al., 2013) through elements of Toulmin's (2003) model, resulting in the creation of the first proposal of an analytical tool. This tool is further elaborated with regard to the concept of justificational mediation (Misfeldt & Jankvist, 2019), a distinct type of mediation in reasoning processes. However, the intricate process that emerges from the analytical tool, governed by the goal of changing the epistemic value of claims, leads to the abandonment of the notion. In paper 3, the analytical tool is further refined and used to define instrumented justification to describe the process of students' justification processes when using an artifact. The analysis concerns a case where a pair of students solve a task designed within the study, the "equal points" task. Some of the findings concern the potentials and challenges of the task. These are further addressed in paper 4, which also describes how the task is further developed. Paper 4 draw on the definition of instrumented justification processes, but the analytical focus evolves around the goal of student tool use, and the potentials and challenges of the "equal points" task for students' exercise of RC. In paper 5, the analytical tool is again utilized for analysis, accentuating the scheme of the scheme-technique duality through the analysis of the scheme's components (Vergnaud, 1998b), elaborating on how students' conceptual knowledge are an integrated component of their instrumented justification processes.

5.3.3 Paper 2 - Developing an analytical tool of the processes of justificational mediation

Paper 2 explores justificational mediation (JM) in the justification processes of a pair of early secondary students using GeoGebra. JM is introduced in the context of computer algebra system (CAS) assisted proofs in textbooks (Misfeldt & Jankvist, 2019). This paper, however, approaches JM in the context of justification processes. The paper aims to expand the understanding of JM by combining Toulmin's (2003) model with the Instrumental Approach (IA) (Rabardel & Bourmaud, 2003) to analyze the process. It emphasizes that JM has the objective of changing the status of a mathematical claim, e.g., from being probable or likely to being perceived as true or false.

The paper proposes reinterpreting Toulmin's model based on the generative and epistemic aspects of schemes (Vergnaud, 2009), a framework for analyzing arguments, to unravel the processes surrounding JM. Using Toulmin's (2003) model amplifies the importance of the qualifier as an indication of the change in the status of a claim, and thus serves as a structure for analyzing JM.

The reinterpreted model of Toulmin is operationalized in the analysis of the justification processes of two 7^{th} grade students, assigned with a task in which they had to predict the movement of two variable points, A = (1,s) and B = (s,1), in GeoGebra's coordinate plane. The examination of the students' informal argumentation, within the digital environment of GeoGebra, dissect the structural components of the JM process in the identification of key structural elements. Specifically, the students generate data as evidence and facts that support their mathematical claims. Moreover, the analysis highlights the critical role of warrants in the students' argumentation, pointing to the inference rules utilized by the students to establish a connection between the generated data and the initial claim.

5.3.4 Paper 3 – Lower Secondary Students' RC in a Digital Environment: The Case of Instrumented Justification

The paper addresses the aspect of justification concerning students' RC within the broader context of the KOM framework, and its implications in a dynamic geometry environment, as processes of instrumented justification. The study highlights the significance of GeoGebra's algebra view in providing symbolic representations alongside graphic representations. However, the potential of dynamic geometry and algebra environments for lower secondary school students remains relatively unexplored. Building on paper 2, the paper presents a revised analytical tool, reinterpreting elements of Toulmin's (2003) model from the perspective of IAME's scheme-technique duality. The aim is to provide insights into the relationship between students' use of a digital environment and their justification processes, shedding light on their RC.

The tool is utilized in analyzing excerpts from two students' efforts at solving a task, where the algebra view's input field is used to construct and transform variable points controlled by a slider. The task given is the "equal points" task concerning the preconstructed points A = (1, s) and B = (s,1). Students are asked to construct a point C, dependent on *s*, so that C and A move on parallel trajectories. Then they are asked: "Can C = B? If so, when?".

Excerpts of the students' justification sub-processes are analyzed, and it is described how the students argued for two opposing claims. The changes in the qualifiers of these claims were analyzed across three justification sub-processes, providing a detailed examination of the instrumented techniques, data, and warrants that the students used to generate and interpret evidence for their claims. The students' use of certain techniques, such as editing the coordinates of points and observing animations, provided insights into their interpretation of data as evidence for their claims, and the warrants they relied on during the justification process.

The use of animations, perceptions of point movement, and the distinction between "colliding points" and "intersecting lines" are also discussed. The discussion relate to the student's interpretation of data as evidence and their gradual recognition of key mathematical concepts toward their evolving understanding of properties and structures inherent in the variable points. The use of phenomenological warrants, such as “the speed of animated points influences when points can be equal” and “the positions and movements of a point are influenced by changing the coefficient of the variable in the coordinate set”, were significant in the students’ justification process. The task caused tension between students’ interpretations of variables as a general number and the need for a generalized conception, particularly in understanding the dynamism and temporality of mathematical objects.

The findings lead to the proposal of a definition of instrumented justification processes: *Instrumented justification is a process through which a student modifies the qualifier of one (or more related) claim(s) using techniques in a digital environment to generate and search for data and warrants constituting evidence for such claim(s).* The theoretical tool for analyzing the IJ processes in DGAE’s like GeoGebra contributes to strengthening the knowledge of students’ RC in a digital environment, shedding light on the intricate relationship between students’ use of digital tools and their justification processes.

5.3.5 Paper 4 - Lower secondary students’ exercise of RC: Potentials and challenges of GeoGebra’s algebra view

Paper 4 dives deeper into the challenges, outlined in paper 3, concerning the “equal points” task. It explores the challenges and potentials of the task for students’ exercise of RC and explains the development and rationale behind its revisions.

The task revision addresses several issues. Firstly, inherent in the task was a prerequisite of viewing variable points as a set of points that can be changed, hindering students’ engagement. Secondly, while slider animation offered phenomenological impressions of movement speed, students required guidance to utilize this functionality. Thirdly, the trace functionality in conjunction with changing the coefficient had the potential for phenomenological impressions in terms of the length of the trajectory. In addition the intersections of traces indicate the coordinate position when points are equal. Fourthly, students needed assistance in manipulating expressions with variables in coordinates. Finally, tasks involving parallel moving points diverted students from geometric property justifications.

The revised task provides context for examining the potentials and challenges for lower secondary students’ exercise of RC toward justifying algebraic properties of variable points. Based on two different class experiments, students’ argumentations are compared between the first and the revised tasks. The students’ work is analyzed from the perspective of KOM (Niss & Højgaard, 2019), and students’ tool use of GeoGebra’s algebra view is analyzed from the standpoint of the IAME (Artigue & Trouche, 2021; Drijvers et al., 2013).

The findings show that the integration of graphic and algebraic representations through sliders in GeoGebra's algebra view holds potential for enhancing students' RC. However, few students successfully related observations from the graphic view to algebraic expressions. Some based their justifications on phenomenological impressions, serving as initial steps of reasoning within the algebraic domain. However, students' struggles to grasp core concepts hindered progress, amplifying the complexity introduced by the algebraic view. Some students who struggle with instrumentalizing relevant techniques in the algebra view may relate to their symbol and formalism competency. Paired with a lack of instrumentation of the graphic view, such students are generally challenged in their exercise of RC.

Finally, the problem-solving strategy that most students implemented was to pick a technique and stick to it. This reluctance to explore other techniques was a challenge for students' instrumentalization of other techniques in the algebra view for both problem-solving and justification.

5.3.6 Paper 5 - Analysing Instrumented Justification: Unveiling Student's Tool Use and Conceptual Understanding in the Prediction and Justification of Dynamic Behaviours

This paper builds on the theoretical framing of papers 2 and 3. Within the context of predicting dynamic behaviors of variable points, paper 5 aims to understand the interplay between students' RC, use of DGAE, and their conceptual knowledge.

A pair of students' prediction and justification of the dynamic behavior of variable points A = (1, *s*) and B = (*s*, 1) in GeoGebra are subjected to analysis, applying the analytical tool for IJ developed in paper 2 and 3. Their process is further analyzed with respect to the conceptual aspects of the scheme-technique duality from IAME (Artigue & Trouche, 2021; Drijvers et al., 2013; Trouche, 2003) in IJ processes. This is done by considering the components of scheme as defined by Vergnaud (1998b), which concern students' personal theorems about concepts, their rules-of-action, and the possibilities for inference toward obtaining goals.

The analysis explores the interplay between data production, interpretation, and progression in terms of techniques. The analysis of warrants provides insights into the progression of students' conceptual understanding as a result of inferences between theorems about concepts, which co-evolve with the progression of techniques. In addition, the progression of instrumental genesis is driven by students experiencing the inefficiency of both rules-of-action and the constraints of the artifact, pertaining to the goal of changing the epistemic status of a claim, ultimately advancing their instrumental genesis. The findings suggest that predicting dynamic behavior can enhance knowledge-based justification, and the progression of technique is driven by students' experience of the inefficiency of techniques and artifacts related to the goal of justification.

The paper also highlights the value of the prediction task, particularly in revealing properties of the variable and challenging students' phenomenological impressions of dynamic behavior. The results emphasize the potential for such tasks to develop a structural conception of the variable and challenge students to move toward theoretically grounded justification. Additionally, the paper provides insights into the progression of students' conceptual understanding and tool use in the context of mathematical RC, instrumental genesis, and inferences drawn between theorems-in-action.

5.3.7 Paper 6 - On the notion of "background and foreground" in networking of theories

Paper 6 stands out from the rest as it relates specifically to the perspective of networking of theories. The paper is a product of the literature review of the co-authors and me. This review was made to identify and understand the research practices and notions of networking of theories. The notion introduced in this paper will be used in chapter 8.

The paper explores the crucial concept of 'background and foreground' theories, which play a pivotal role in the networking of theories within the context of mathematics education research. Through a hermeneutic literature review, we analyze how the notions of foreground and background theories are utilized in the literature on networking theories. We support our analysis with two cases that illustrate the relative and absolute distinctions of these terms, providing concrete examples for our discussion. The absolute perspective considers foreground and background theories distinct and fixed categories, as theories that stem from inside or outside mathematics education research, with clear delineations between them. In contrast, the relative perspective views foreground and background theories as more fluid and context-dependent, allowing for their roles to vary depending on the specific research context or situation. The study highlights the coexistence of both relative and absolute distinctions in the literature and discusses the implications of each perspective. While the relative distinction can cause unnecessary confusion in terminology, it also offers a nuanced understanding. Based on these findings, we propose a novel concept, 'framing theories', which we believe can effectively address the nuances of background theories within and outside mathematics education research.

6 THE DESIGN AND CONSTRUCTION OF TASKS

In this chapter, I address RQ1:

> In what ways can tasks be designed to encourage lower secondary students to exercise their reasoning competency when using a dynamic geometry and algebra environments in the case of justification focusing on variables as a general number?

This chapter emphasizes the study's design process. It contributes design principles that are justified through theoretical perspectives used in the design and construction of tasks and the retrospective analyses of each iteration. I first introduce and provide rationales for the HDH (Prediger, 2019) and their development into design principles (Bakker, 2018). The design of learning environments and the construction of tasks require both an analytical and a design perspective (McKenney & Reeves, 2018). In section 6.1, I describe the analytical perspective that underpins the complete design process. As I first present the analytical perspective, it may appear to be a priori analysis. In reality, the analytical perspective has formed alongside design processes and classroom experiments but is described and argued for in separate sections to convey a clear picture. The theoretical development that intertwined with the design processes is described and discussed in Chapters 7 and 8.

Following, I report on the design processes throughout the three iterations by describing impactful strides of the design and task construction processes and the development of design principles. Section 6.2 explicates the initial explorative design process in iteration 1. Based on the explorative processes, a microworld of variable points and related tasks is developed in iteration 2, which is described in section 6.3. Progression in iteration 3 is elaborated in section 6.4. In each iteration, the retrospective analyses summarize and report on results and observations that have materialized in the development of specific tasks. I do not present a complete picture of the retrospective analysis and the affiliated design processes, as this is beyond the limits of the thesis. The intent is to outline analysis and observations that give context to the included papers.

Finally, in section 6.5, a comprehensive discussion of the educational design processes is presented. This includes a detailed exploration of the design principles that have evolved from the initial HDH, providing a comprehensive understanding of the theoretical and practical aspects of the design process.

6.1 THE ANALYTICAL PERSPECTIVE - FOUNDATIONS FOR DESIGN

The analytical perspective considers different normative theory elements in the design process that elaborate and justify aims and principles (Prediger, 2019). First, I discuss designing tasks for RC and justification, then designing for using tools and representational issues of GeoGebra concerning variables as a general number, and finally, the content concerning the age group of lower secondary students. HDHs are formulated from the analytical perspective and revisited from the design perspective.

6.1.1 Designing for students' exercise of RC

Designing for students exercise of RC involves the construction of mathematical problems that allow students to produce and justify claims and solutions when using GeoGebra.

In reasoning, students often struggle to identify the relevant concepts and properties of a problem (Duval, 2007); Lithner (2008) adds that this can be related to students anchoring their reasoning on surface properties instead of the intrinsic properties of a given problem. He gives the following example: "In deciding if 9/15 or 2/3 is larger, the size of the numbers (9, 15, 2, 3) is a *surface* property that is insufficient to consider while the quotient captures the *intrinsic* property" (Lithner, 2008, p. 261). On the one hand, for students to be able to anchor their arguments in intrinsic properties, they must at least be aware of them and know when they are relevant to, in the words of KOM, be insightful and respond appropriately to the challenge (Niss & Højgaard, 2019). The design and the problems must therefore be based on the student's existing knowledge and competency. On the other hand, the problems should not be trivial, as students can just apply well-known routinized techniques or algorithms already considered trustworthy (Lithner, 2008). Familiarity with a mathematical task can cause the students to not justify their claims and solutions, as the need for validation diminishes. This observation aligns with the description of mathematical problems within the KOM framework, where a mathematical problem can only be considered as such if it poses a challenge to the individual attempting to solve it. This rationale forms the following HDH:

A: The *intrinsic* properties of a posed problem must be familiar to the students, but the type of problem posed should be novel to students (based on Duval, 2007; Lithner, 2008; Niss & Højgaard, 2019).

This also implies that some progression within tasks and content is necessary so that students keep a sense of novelty.

The second HDH is inspired by the work of White and Gunstone (1992), who proposed the task structure "prediction-observation-explanation". This structure involves students making predictions about the outcome of an event, justifying their predictions, and subsequently testing their predictions through observation. White and Gunstone suggested this structure for teaching and learning in the natural sciences, but it has also been successfully adapted for task design in mathematics education. Research on prediction tasks in mathematics education has demonstrated that requesting students to predict outcomes can encourage them to engage in mathematical reasoning based on their conceptual knowledge (Kasmer & Kim, 2011; Lim et al., 2010). I emphasize the justification of the prediction and refer to White and Gunstone's task structure as "justified prediction-observation-explanation". The task structure is particularly relevant in digital environments, where the instant feedback allows students to test mathematical conjectures and claims empirically, akin to experiments in natural science, where the observed may confirm or refute a prediction or need further explanation. Researchers such as Olsson (2017) and Højsted (2021) have applied this structure particularly to GeoGebra, embedding it within a digital environment. In the case of using commands in the algebra view, the justified predictions concern the translation between algebraic and graphical representations and the dynamic behavior of the constructed objects. This rationale forms the HDH:

B: "Justified prediction-observation-explanation" tasks (White & Gunstone, 1992) can support students in forming claims and engage in justification processes about algebraic relationships and concepts based on their own knowledge (Kasmer & Kim, 2011; Lim et al., 2010).

6.1.2 Designing for the use of digital tools

The IAME determined that learning to use a tool is a lengthy process specific to certain situations. This is why even experienced users can face challenges when confronted with a new problem. It is essential to consider this aspect when designing tasks that involve digital tools. When introducing students to a new tool, such as functionalities of the algebra view, it is essential to guide them through the process of learning how to use it. Some challenges can be related to variations in syntax across different regions or languages. For example, in Danish a comma is used instead of a dot for decimals. The students will have to have to be reminded of this. Some tools, like a slider, may be completely unfamiliar to students, and some guidance on its use is needed for particularities. For example, in addition to pulling the slider, which many students do intuitively, the value of the variable can also be altered by clicking and typing in a specific value. This can be useful if students want to test a specific value.

In the literature review (paper 1), I found a scarcity of examples or discussions of students' use of the tools in GeoGebra's algebra view or of papers discussing GeoGebra's algebra view concerning students' reasoning, with respect to variables as general numbers within the age group. Only five conference papers were included:

> The five identified studies are all peer-reviewed but cannot be perceived as the same quality as a journal paper. This indicates that the research on the potentials of GeoGebra's Algebra View and its functionalities for mathematical tasks and processes other than functions is still developing. Two of the papers are theoretical, while three

> present empirical results. Four of the five studies make use of GeoGebra, and one study by Lagrange and Psycharis (2011) makes use of a programming "turtle world" software, LOGO, which has very similar affordances to that of the Algebra View in GeoGebra. The software uses a programming language, whereas the Algebra View in GeoGebra uses standard algebra notations and commands specific to the program. (Gregersen, 2022, p. 4)

A literature review by Yohannes and Chen (2021) on GeoGebra in MER draws a similar picture. They find that very few studies concerning this age group have been published in journals, and those identified focused on geometry, while other mathematical domains concern higher educational levels:

> The result of this study indicated that, among all of the studies, Geometry accounted for the highest number, with a total of 11, followed by analysis (n = 9) and Discrete Mathematics/ Algebra (n = 5); there were no papers in applied mathematics and general/ foundational mathematics. It can be realized from this finding that the content domains of mathematics education were mainly senior secondary school mathematics and higher education mathematics (Yohannes & Chen, 2021, p. 7).

Consequently, in the study's first DR iteration, the design and construction of tasks were predominantly influenced by the creative perspective to discover affordances of the algebra view with educational value toward students exercising RC concerning variable as a generalized number. In the following, I discuss the representational infrastructure of GeoGebra, in this case with regards to students' use of sliders and algebraic expressions in the algebra view. A dominant issue with the representational structures of GeoGebra is that the complexity and amount of information may hinder students' exercise of RC, as they do not have the conceptual knowledge to understand the representational system. In many cases, the algebra view contains symbols referring to mathematics unknown to students in lower secondary education and a vast amount of information for the student to manage, even in a quite simple construction. This is exemplified in Figure 12 where a dynamic circle with a line representing its radius is constructed in GeoGebra. The construction in the algebra view comprises five entries, each representing an object in the graphic view. One entry contains the equation of a circle, and in all, the five entities contain 13 letters, 11 numeric values, and four words.

All in all, this is a lot of symbolic information to process. In addition, the equation of the circle is unknown to students in lower secondary school.

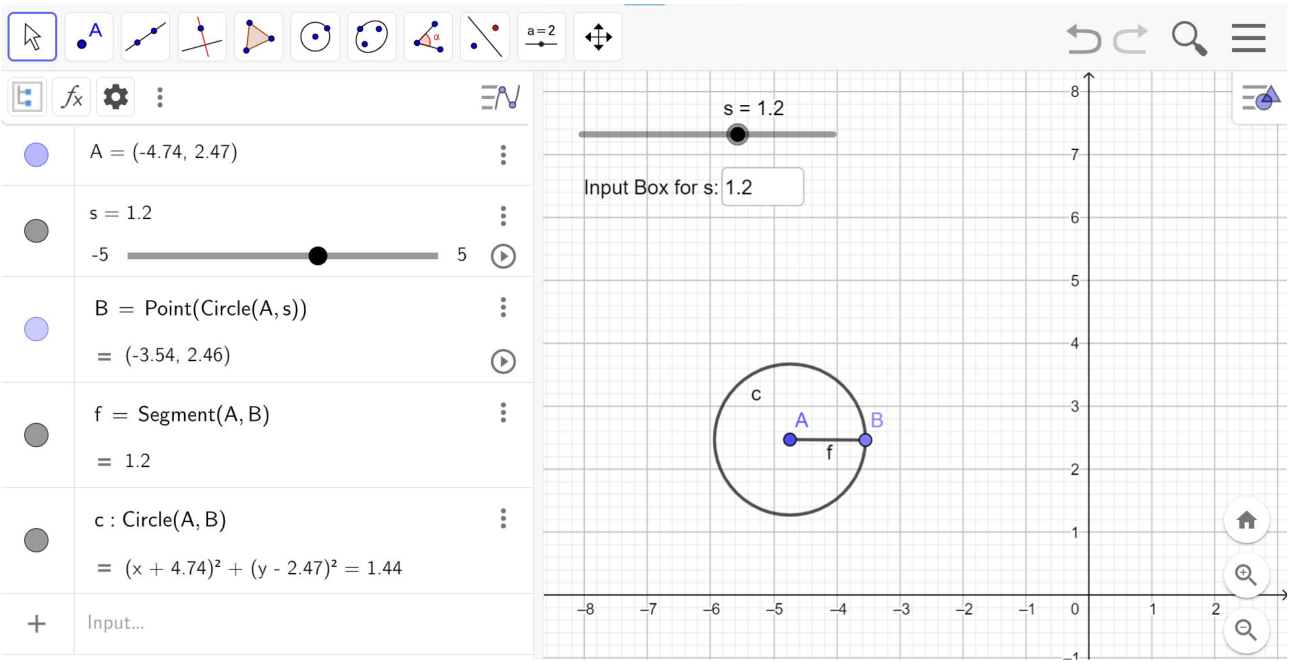


Figure 12 – The construction of a dynamic circle and its radius, illustrating the vast and complex information in the algebra view

The second finding concerns the use of sliders in GeoGebra. Sliders have the potential to link geometric and algebraic representations (Mackrell, 2011). Only a few examples of this exist concerning the variable as a generalized number. The slider can be used to search for numerical cases by evaluating geometric configurations, as in Tanguay et al. (2013), where students explore the number of some polygon that covers a surface to identify divisors. In Soldano and Arzarello (2017), students must investigate the numerical circumstances by which two circles become tangent by manipulating three sliders that control the radius of each circle and the distance between the circles. In these examples, the students do not have access to the algebra view but only to sliders on the graphic view. This limits the students to making conjectures about the numeric values displayed, not the relationships expressed in algebraic relationships (Paper 1).

To explore the potential of having access to symbolic representation in combination with the slider and graphic representation, we need to expand beyond GeoGebra, to other uses of variables, such as in functions or equations.

The MiGen project, explored by Noss et al. (2012) and Mavrikis et al. (2013), focused on students learning pattern construction through the "expresser" microworld. The project involved a box to represent numerals as a generalized number, with students employing it to express variable relationships in constructed patterns. The crucial step was to provide students with graphic representation experiences for constant values and variables in generalizing numeric patterns into algebraic relationships. “Locked” and “unlocked” boxes were utilized, and the box itself became a symbol enabling explicit representation of changed and unchanged elements, as well as perceived relationships within a model.

In another experiment with the microworld MoPiX, 17 year-old students could modify equations of bouncing ball graphs, altering animations such as making balls collide or ‘juggle’. Kynigos et al. (2010) observed students progressing to investigate how variables and constants influenced the graphs, forming connections between MoPiX equation syntax and the animated behaviors of objects.

Two distinct approaches emerged in these two projects: the first used graphic representation to enhance variable understanding, while the second introduced symbolism and prompted students to explore its influence on graphic representation.

The two studies indicate that a microworld where students create and manipulate symbolic representations that are simultaneously graphically represented serves as a context for conjecturing intrinsic algebraic properties related to variables. However, as the representational structures of GeoGebra can be complex, the representations must be considered with concern to the age group.

The following HDH includes considerations of designing tasks for the use of the algebra view:

C: Tasks where students create and manipulate symbolic representations that are simultaneously graphically represented can serve as a context for conjecturing about intrinsic algebraic properties related to variables (Kynigos et al., 2010). However, the complexity of the representational infrastructure of GeoGebra must be considered so that the representations students are to handle are accessible to them (paper 1).

6.1.3 Designing for lower secondary students in Denmark

In this section, I discuss the mathematical content with regard to the concept of variables and suitable intrinsic properties for the age group of lower secondary students by taking into account the HDHs A and C.

A. The intrinsic properties of the task must be known to the students but in problems novel to students (Duval, 2007; Niss & Højgaard, 2019), because it fosters a need for validation (Lithner, 2008).

C. Tasks where students create and manipulate symbolic representations that are simultaneously graphically represented can serve as a context for conjecturing about intrinsic algebraic properties related to variables. However, the complexity of the representational infrastructure of GeoGebra must be considered so that the representations students are presented with are accessible to them (based on review).

Specifically, I discuss what concepts are familiar to students in 7th grade and can be symbolically and graphically represented in GeoGebra, furthermore, in such a way that students can construct and manipulate them within the algebra view.

Variables are one of the most fundamental concepts of algebra (Knuth et al., 2011). They have different meanings and uses, such as representing a specific unknown value or function, a generalized number that can represent several or infinite values, or a variable in a functional relationship where it represents a range of values and has a systematic relationship to another set of values (Rhine et al., 2019). This study considers the variable as a generalized number, which means that it represents a range of values or all possible values.

It is widely recognized that concept formation has a dual nature and a development involving processes and objects (Douady, 1991; Dubinsky, 1991; Noss et al., 2009; Sfard, 1991). The early stages of mathematical education will typically focus on comprehending basic processes, such as counting or performing calculations. For young students, concepts are initially tied to processes within specific numeric situations. However, as students move on to lower secondary education, they must develop a structural understanding of these processes (Sfard, 1991). Ideally, concepts evolve into abstract objects, enabling the exploration of structures and relationships (Douady, 1991).

Concerning variables, Noss et al. (2009) emphasize that generalization involves moving beyond the specific, recognizing the structural properties, relationships, and patterns that variables (and constants) represent. Introducing variables often marks students' first step into objectification, requiring them to perceive a letter as representing all values subject to the same computational manipulation as numeric values. This means that students must gradually objectify the processes into abstract and structural concepts that can be manipulated. The variable represents relationships and properties in structures with other objects. Hence, as objects in algebraic expressions, they represent general rules that can be deduced from patterns or families of problems (Rhine et al., 2019). Handling variables requires students to recognize the type of variable applied in a specific context and how it relates to other objects. 7th grade students in Denmark have typically been introduced to the definition of a variable as an expression of all values, and they have experienced procedures concerning variables in equations, functions, and formulas (Ministry of Children and Education, 2019).

As argued above, the complex representational system that commence from the use variable in GeoGebra's algebra view can preclude students' engagement with justification. To keep the representation manageable, I take the position that the representation should be as basic as possible to let students focus on the problem and justification rather than having to grapple with the representation system.

In GeoGebra, as in both Euclidean and Cartesian Geometry, the *point* is the most basic object, followed by lines and segments. Cartesian Geometry is taught in Danish primary education from the early grades onwards (Ministry of Children and Education, 2019). Representations from the domain of Cartesian Geometry are also incorporated in the representational infrastructures of GeoGebra, making a basic concept like points in the coordinate plane accessible for students to represent and transform algebraically. Plotting coordinates into a coordinate system is, for the most, a trivial task in the 7^{th} grade. From the 7^{th} grade onwards, students are introduced to linear and non-linear equations and functions, including the coordinate system's graphical representations. They are, however new concepts for the students and cannot be considered well-known mathematical knowledge at that time.

Intrinsic properties and concepts can be algebraically and graphically represented by points and lines and can be, depending on an explicit variable, equality, infinity, limits, parallelism, length, and distance. Points can also have intrinsic properties, as vertices in a construction, or indicate a specific property of other objects, e.g., the midpoint of a line.

What concrete properties and concepts to pursue are explored further from the creative perspective.

6.1.4 Overview of the humble design heuristics

The HDHs for the first iteration to be refined in the following are:

> If you want to design tasks for early secondary students to exercise their RC in justification about variable as a general number when using GeoGebra's algebra view and graphic view, you are advised that:

A. The intrinsic properties of the task must be known to the students but in problems novel to students (Duval, 2007; Niss & Højgaard, 2019), because it fosters a need for validation (Lithner, 2008).

B. "Justified prediction-observation-explanation" tasks (White & Gunstone, 1992) can support students in forming claims and engage in justification processes about algebraic relationships and concepts based on their own knowledge (Kasmer & Kim, 2011; Lim et al., 2010).

C. Tasks, where students create and manipulate symbolic representations that are simultaneously graphically represented can serve as a context for conjecturing about intrinsic algebraic properties related to variables (Kynigos et al., 2010). However, the complexity of the representational infrastructure of GeoGebra must be considered so that the representations students are to handle are accessible to them (paper 1).

Other didactical concerns that should be considered in the construction of tasks are:

- Include introductions to the new tools and support the interpretation of the representations they produce in the support of students' instrumental genesis (IAME).
- Progression in complexity

6.2 ITERATION 1: EXPLORING DESIGN POSSIBILITIES

The following section describes the creative perspective of the first iteration and presents snapshots from the retrospective analysis, which leads to a revisit of the HDH.

6.2.1 The creative perspective - exploring design possibilities

The creative perspective in the first iteration was characterized by constructing tasks concerning a range of intrinsic properties and representations to see how students would manage. In these processes, I also decided to focus solely on tasks that used the algebra view, as it became progressively more evident from the analytical perspective that this was a less explored territory in MER for the age group. At this stage of the design phase, the aim was to explore possible design ideas.

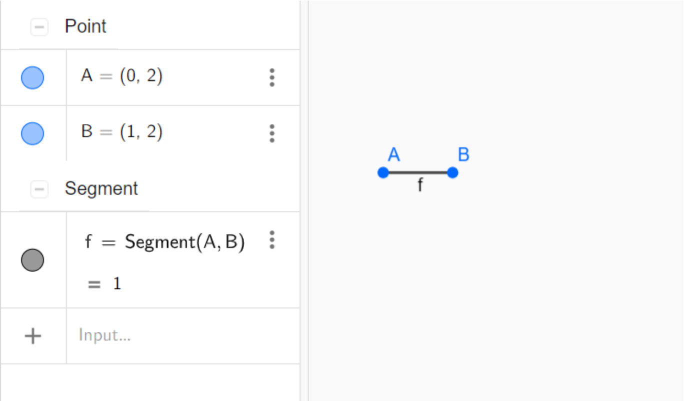
Seven tasks were part of the explorative design and became part of the classroom experiments in the first iteration. In the following retrospective analysis, I discuss the lessons of these experiments.

6.2.2 Snapshots from the retrospective analysis

Out of the seven tasks, this section discusses three of them, representing core issues observed in the classroom experiment and considered in the retrospective analysis. Two tasks had issues that prevented their further development, while the third task inspired the creation of a microworld with *variable points*.

The first issue was tasks where the algebraic properties were not represented explicitly, which led the students to focus more on the geometric properties represented in the graphic view. The task "Relationships between lines" (Figure 14) exemplifies this.

Title of task: Relationships between lines



Questions

For the tasks, you only have access to the algebra view and the graphic view. In the app, you will find the line segment AB = *f*.

1. Construct line segment CD by typing in the input field:

   C = (x(A),0)

   D = (x(A)+3AB,0)

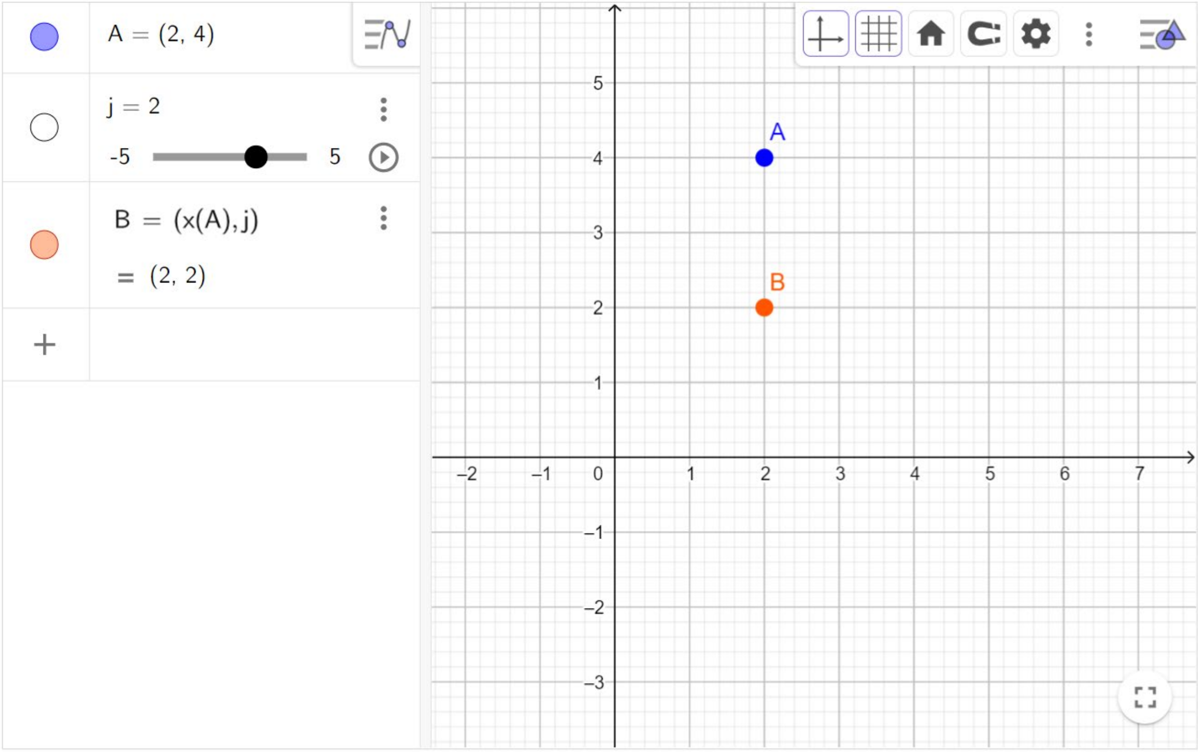
   Segment(C,D)

2. Investigate and explain what happens when you drag A or B.
3. Is it true that CD is always 3 times AB? Justify your answer.
4. Construct a line segment EF that is always half of CD.
5. Justify why your construction is correct.
6. What is the relation between line segments AB and EF? Justify your answer.
7. Is it possible to construct a line segment four times the length of EF that is also double the length of AB? Explain and justify.

Figure 13 – The task "Relationship between lines". Above: The GeoGebra app for the task. Below: the questions posed in relation to the task

This task made me aware of how variables can be constructed and represented in GeoGebra. Recall that dynamic construction is achieved using variables, which can be either implicitly represented through geometric shapes or explicitly represented using the slider tool. In the "Relationships between lines" task, the variable was represented implicitly, which appeared to make the students focus more on geometric properties and visual arguments, e.g., comparing lines by visual assessment.

Title of task: Construct a dynamic right-angled triangle



Questions

1. Explore how points A and B move as you change *j* by pulling the slider and dragging the points.
2. Explain how point B moves and why.
3. Construct a new point, C, that has the distance *j* to point A, and ABC is a right-angled triangle, also when you change the value of *j*.
4. Justify why your solution still works as you change the value of *j*.

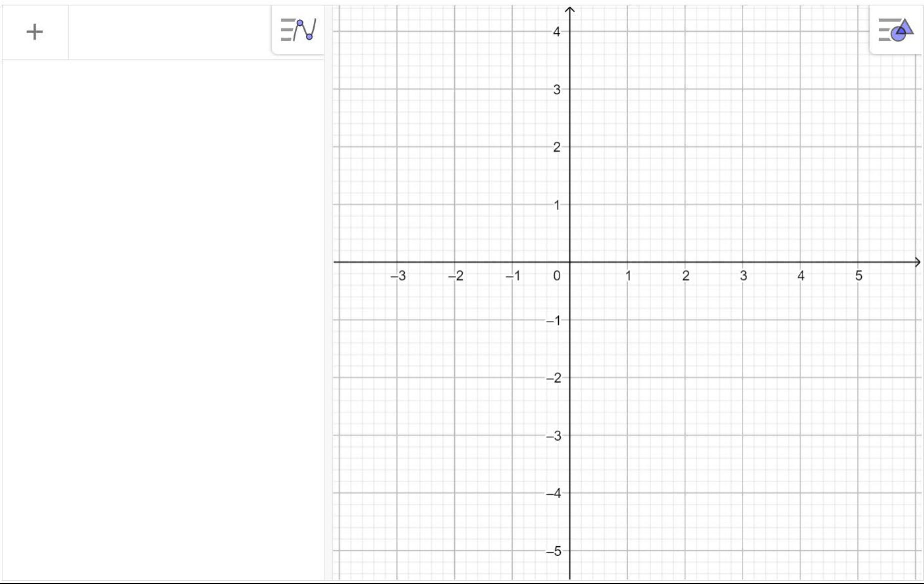
Figure 14 – Construct a dynamic right-angled triangle. Above: the GeoGebra app for the task. Below: the questions posed in the tasks

The second issue pertained complexity of representation when using symbolic notation in the algebra view. Both the "Relationships between lines" (Figure 13) and “Construct a dynamic right-angled triangle” (Figure 14) exemplify this concern.

Notice that in both tasks, the endpoints of new lines depend on the position of already constructed points. For instance, C = (x(A),0) and D = (x(A)+3AB,0) are dependent on the x-value of the point A. This poses two problems.

The “Relationships between lines” task involves representations from three distinct domains: coordinate geometry (since line endpoints depend on the coordinates of other endpoints), arithmetic (as multiplicative properties are represented), and geometry (in the definition of lines). The students engaged with the tasks but could get lost trying to decipher how to construct new points and lines. Some students did not recognize that they were typing in coordinates. Instead, they tried to copy the

Title of task: Moving points



Empty GeoGebra applet with access to the algebra view but no geometric tools.

Question 1 is solved with paper and pen on a coordinate system

B = (1,*s*) and C = (*s*,1),

1. Show on a paper coordinate system how you think B and C move as *s* changes value.
2. Explain why.
3. Type points B and C into the input field of the algebra view.
   a) change the value of *s* by dragging the slider.
   b) turn on the trace of B and C (by right-clicking on the points).
   c) again, change the value of *s* by dragging the slider.
   d) explain what you see.
4. Did the points move as you predicted?
   If not: Explain how they move and why. What did you miss in your prediction?

If they did: explain what you understood about the points that made your prediction true.

Figure 15 – Moving points. Above: The GeoGebra app for the task. Below: The questions posed in the tasks

systemic of the existing objects, which can be considered imitative reasoning (Lithner, 2008), and they would base their justification on visual assessment alone.

In the “Construct a dynamic right-angled triangle” task, the surface property is the geometric properties of a right-angled triangle, and the intrinsic property is that a variable represents the same value every time it appears with in the same task. Both properties were familiar to the students in the pilot test, and some would consider the variable, while others would get confused by the notation of interdependent points.

In both tasks, the students had to learn how to interpret the new notation by exploring the representation. The justification focused on the representational structure rather than the intrinsic properties of the tasks. The retrospective analysis indicated that the use of notation where points depend on other points appear too unfamiliar to the age group.

The “Moving points” task (Figure 15) follows the “justified prediction-observation-explanation” structure. It later became the outset for the design idea of *variable points*, leading to the design of a microworld (elaborated in the following sections). This task was more accessible to the students as the ordered pairs in the algebra view were known to them, and the use of variables was explicit and simple with familiar algebraic notation. The graphical representation of a point is also simple and familiar. The “simplicity” I hypothesized would allow students to explore and put forward conjectures to be justified, rather than getting “lost in translation” of complex notation.

6.2.3 Revisiting the HDH

The retrospective analysis of the explorative design adds specifications to the HDHs A and C, which have been added in bold:

If you want to design tasks for early secondary students to exercise their RC in justification about variable as a general number when using GeoGebra’s algebra view and graphic view, you are advised that:

A. The intrinsic properties of the task must be known to the students but in problems novel to students (Duval, 2007; Niss & Højgaard, 2019), because it fosters a need for validation (Lithner, 2008). **Intrinsic properties expressed in terms with variables may be signified using explicit variables rather than implicit ones. This is because implicit variables may cause students to focus on geometric properties rather that algebraic properties and that explicit variables provide students with access to direct manipulation of terms with a variable in the algebra view.**

B. “Justified prediction-observation-explanation” tasks (White & Gunstone, 1992) can support students in forming claims and engage in justification processes about algebraic relationships and concepts based on their own knowledge (Kasmer & Kim, 2011; Lim et al., 2010).

C. Tasks where students create and manipulate symbolic representations that are simultaneously graphically represented can serve as a context for conjecturing about intrinsic algebraic properties related to variables (Kynigos et al., 2010). However, the complexity of the representational infrastructure of GeoGebra must be considered so that the representations students are to handle are accessible to them (paper 1). **This might be obtained with variable points as ordered pairs and simple algebraic expressions familiar to lower secondary students. In addition, using notations of points depending on other points is not advisable for lower secondary students.**

6.3 ITERATION 2: THE MICROWORLD OF VARIABLE POINTS

From the creative perspective, the following elaborates on the design of a microworld and the construction of a task sequence. Then, as in the previous iteration, snapshots of the retrospective analysis are elaborated, and the HDHs are now progressed into design principles.

6.3.1 The creative perspective: From explorations to the design of a microworld

The idea of *variable points* grew from the task “Moving points”, presented above in Figure 15, but was still unexplored as a learning object for justification. It was somewhat inspired by visual programming, such as JavaScript – that is to say, making things move on a screen through computational tools. Programming can engage children in reasoning about structures and patterns in computer algorithms and is increasingly incorporated into mathematics curricula (Kilhamn et al., 2022). Nonetheless, as a mathematics teacher I encountered issues regarding programming in mathematics education, which is also reflected in MER. Programming has its own syntax, which is not directly transferable to mathematics thoery. For example, a variable in mathematics is an expression of generality, whereas in programming, a variable stores a specific value that can be changed under certain events (Bråting & Kilhamn, 2021; Kilhamn et al., 2022).

Another example is the definition of space. In most programming languages, including JavaScript, the plane is described in pixels in positive integers; (0,0) is in the top left corner of a screen or window, and pixels describe length and width relative to the screen size. This is considerably different from the coordinate plane, which is endless in two dimensions from its origin. Programming activities in regular programming environments lack mathematical theory and do not necessarily influence students’ mathematical capabilities (Benton et al., 2017; Boylan et al., 2018;

Kilhamn et al., 2022). Hence, Kilhamn et al. (2022) argue that programming activities in mathematics should draw on mathematical theory rather than computer theory if the aim is to support mathematical development.

From this perspective, GeoGebra provides a *mathematical* programming language, and the microworld of v*ariable points* is an attempt to "make something move on the screen" that embodies a mathematical subdomain where students can engage in justification. Though programming inspired the idea of variable points, it had to be developed into something tangible for the student to engage with and exercise their RC. Therefore, I will leave the programming analogy and describe the design as a microworld.

There are different approaches to designing tasks for digital tools. Frequently, traditional math problems have been redesigned to exploit the affordances of a particular tool. This study follows another design tradition of microworlds, initially envisioned by Papert (1980), who aimed to reconstruct mathematics teaching using computer software as media designed for mathematics learners. Microworlds are now recognized as computational environments incorporating a cohesive set of scientific concepts and relationships. They are thoughtfully designed to allow students to engage in exploration and construction activities that foster meaningful learning experiences, aided by carefully curated tasks and pedagogical strategies (Healy & Kynigos, 2010; Sarama & Clements, 2002). They provide students with an environment to analyze the components of objects and construct and deconstruct objects. This should facilitate the exploration of mathematical relationships among and between the objects and their corresponding representations (Hoyles, 1993). Since Papert (Healy & Kynigos, 2010; Papert, 1980) introduced microworlds, it has transcended mathematics education into the natural sciences, and it has been used to design many digital technologies.

Commonly, the structural aspects of a microworld are a set of computational objects designed to reflect the structure of mathematical entities within some subdomain of mathematics. It does so through (often new) multi-representations that link the underlying mathematical or scientific entities or objects. Typically, it has a symbolic and a graphic component, but it can have others. The objects and operations can often be combined to form more complex objects or operations. Furthermore, a microworld will usually include a set of activities that support students in examining the structure of the microworld, e.g., worksheets or verbal instructions in which the student is challenged to use the objects and operations to solve a given problem (Edwards, 1998).

The design process was characterized by constructing tasks and showcasing them in different settings to get feedback from colleagues, supervisors, professors at PhD courses, and the participation teachers. In the following, I present the microworld and two selected tasks.

The central objects of the microworld are variable points, which embody variables as a generalized number in its singularity, and simple algebraic expressions (e.g., with a factor and/or a coefficient) with coordinate geometry as the graphic representation. In its most simple form, the microworld of *variable points* appears like this:

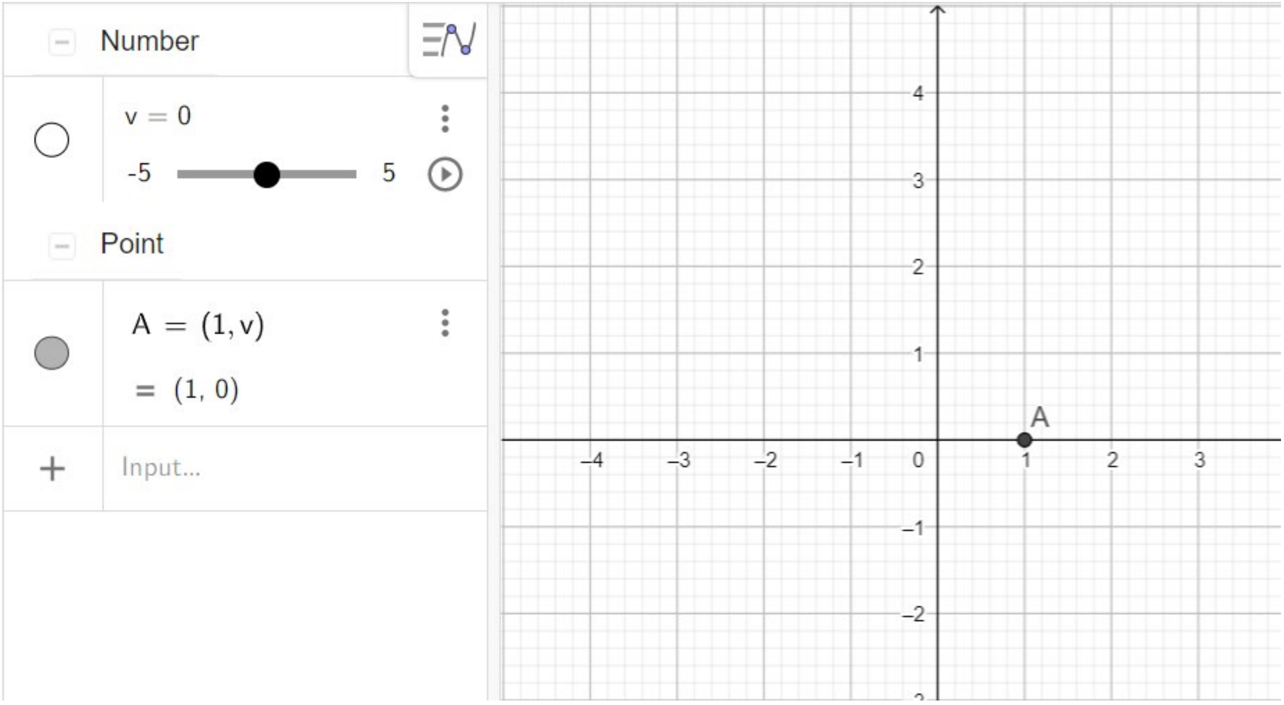


Figure 16 – Basic state of the microworld of *variable points*

In the algebra view, there is an ordered pair with an explicit variable. In the graphic view, the ordered pair is represented as a point in the coordinate plane; the point changes its position as the value of the variable is changed with the slider. The variable is restricted to [-5,5], creating a limited space of the dynamic movement, and points can be "shifted" in either dimension by adding a constant in the ordered pair. There can be several variable points, and the ordered pair can contain simple algebraic expressions and have a variable in both the x and y coordinate. The primary artifacts are the slider tool and constructing ordered pairs through the input field. The two-dimensional space of the coordinate system allows students to justify properties and relationships of both variable and constant terms in the dynamic movement of points. The dynamic representation allows students to consider both the structural and variable aspects within the same low-complexity representation. If students can relate the movement of the points to the terms in the algebra view, the microworld allows students to make claims and justify algebraic properties. This means that students working within the microworld through retrospective analysis can advance HDHs A and C into more concrete, elaborate design principles concerning justification. Conversely, HDH B requires the implementation of the "justified prediction-observation-explanation" structure.

Construction of a tasks sequence

For the classroom experiments in iteration 2, I constructed a task sequence organized into three main problem sets, with several tasks and sub-questions. Furthermore, the task questions were formulated for students to exercise justification. An online version was implemented in Class A. In Class B and C, a version with a Word document and online GeoGebra apps was implemented. Both versions, including task sequences, can be found in Appendix A. The three problem sets progress in complexity. The first set is introductory and does not involve any variables. Instead, it reminds students of their knowledge of ordered pairs and points in the coordinate system. In this set, students are asked to conjecture and justify the relationship among static points. The second problem set progresses to variable points in one dimension, either in the x or y value of the point. The third problem set involves variable points in two dimensions, with x and y values.

Throughout the sequence, tools available in the geometric toolbar is kept to a minimum to ensure students use the algebra view to complete the tasks. The introduction to the tools, such as the slider and the trace function, was integrated into the tasks when needed.

Teacher collaboration

Initially, I met with the teachers to go over the task sequence and adjust specificities according to their students and the class discourse. In addition, this allowed both the teachers and me to support the students as they worked through the task. I supplied the teachers with a teacher instruction guide, including suggestions on how to guide the students without giving them specific answers and how to support them in forming justifications and solutions to each subtask. Between each classroom experiment, the design was evaluated together with the classroom teacher. In class, we noticed that students struggled to formulate written answers, even when they were able to explain and justify orally. To support students' written justification, we therefore developed an answer guide, which was introduced to the students at the start of the experiment.

Below is an example of an answer guide for question 4b in problem set 3 (Appendix A). The answer guide was inspired by Duval's (2007) method of structuring mathematical reasoning with two premises and a conclusion, but recognizing that students are not required to produce formal proofs but justifications. Therefore, the answer guide prompts students to elaborate on their mathematical knowledge about relevant intrinsic properties and connect it to their answer. The answer guide meant that tasks were now presented in a Word document, rather than being online.

Answer:	Answer guide: Your argument must include the following points: • What is required for a point to move from the 2nd quadrant to the 4th quadrant? • What is the coordinate set for J? • Why does it lead to that J moves from the 2nd quadrant to the 4th quadrant?

The evaluation with teacher 1 after the test in classroom B concluded that students still struggled to form cohesive written arguments, but the answer guide prompted students to further reflect on their solutions and incited them to refer to intrinsic properties. Consequently, it also added to the time students spent on each task, thus fewer students completed the whole set. Going forward, to maintain focus on the student's use of the algebra view, the impact of the answer guide is not further explored, but I consider it a variable in the design that enhances the data obtained.

6.3.2 Snap shots from the retrospective analysis

As Edwards (1998) points out, students' difficulties with a microworld can arise from issues in the design and interface and can be helped through changes to design and tasks. Other difficulties are evidence that the student is confronting a significant learning opportunity. Indeed, the retrospective analysis should identify tasks that hindered students from exercising their reasoning competencies in the justification process. For task development, the retrospective analysis explores how the microworld allows and hinders opportunities for students to engage in justification processes as an exercise of RC.

To gain a comprehensive understanding of how students approached the tasks, and to identify the significant tasks, each data set (representing the work of one student pair) was compiled into a collective Excel sheet. I summarized the students' responses and noted any interface issues. I identified and analyzed cases of students appropriating the microworld in their justification processes and confronting significant learning opportunities concerning properties of the variable. An analytical tool was developed in iterative processes as an attempt to capture the intricacy of the interplay of students' tool use and justification processes. The development of the analysis tool and results of these analyses are elaborated in Chapter 7 and 8. Cases where students were challenged but engaged in reaching a solution and justifying it provided deeper insights on the microworld and task design, as they both exposed challenges within the design and how affordances of the microworld were (intendedly and unintendedly) used by the students.

Two tasks emerged as particularly significant: the prediction tasks and the "Equal points" task. Both are presented below. Parts of these two tasks also appear in the papers 2, 3, 4, and 5, but the full versions presented here can afford additional context. Following the elaboration of iteration 3, I present some of the design development that emerged from analyzing students' work on these two tasks.

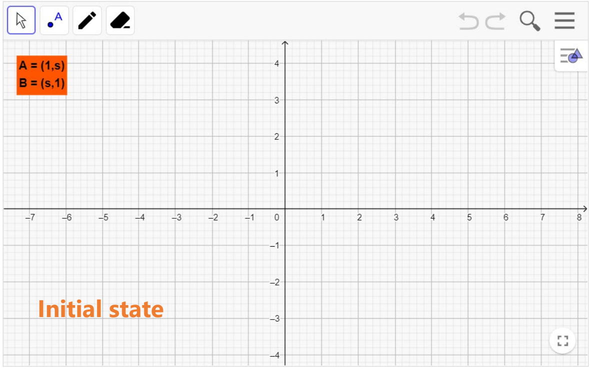
Prediction tasks

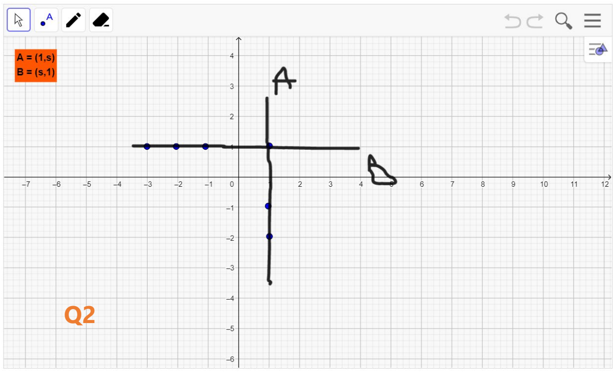
The "justified prediction-observe-explain" task structure was implemented in the task sequence as the first tasks in problem sets two (concerning one-dimensional variable points) and set three (concerning two-dimensional variable points). Below are the two prediction tasks illustrated in Figure 17 and Figure 18. The first prediction task is discussed in paper 2 and 5. The two tasks were developed from the "moving points" tasks (see Figure 15). As pertained in paper 5,

> "Unlike traditional positions of prediction in mathematics education as a statement or conjecture anticipating either the solution to the problem or the strategy used to reach a solution (e.g., Boero, 2002; L. A. Kasmer & Kim, 2012; Palatnik & Dreyfus, 2019), the intention in this case is to leverage predictions and thus give "students the opportunity to defend or refute ideas" (Kim & Kasmer, 2007, p. 298). Consequently, I consider the prediction task a problem in itself…" (p.5).

A notable modification compared to the explorative version in iteration 1 (see figure 16) is the shift from making predictions on a traditional paper coordinate plane to a restricted interface of GeoGebra. The confined interface transforms the graphic view into a notation interface with dynamic tools that allow the students to move points on the screen and trace their movements, replicating dynamic movement. The restrictions are enforced to prevent students from constructing the points by typing the variable points into the algebra view. Predicting dynamic behavior and justifying the prediction allows inference that would not be possible in a pen-and-paper environment, as both the constant and the variable would be represented statically (Noss et al., 2012).

Titel: “Justified Prediction- Observation-Explanation” task of one-dimensional variable points





Q1

Read:

Points can have a variable in the coordinate set such as these two points:

A = (1,s) and B = (s,1) where *s* is a variable.

Q2

Show and explain how you think points A and B move in the coordinate system when *s* changes value.

(To do so, you can use the tools in the toolbar, and you can also right-click and use the tools there)

Q3

Justify your hypothesis - why do A and B move as you claim?

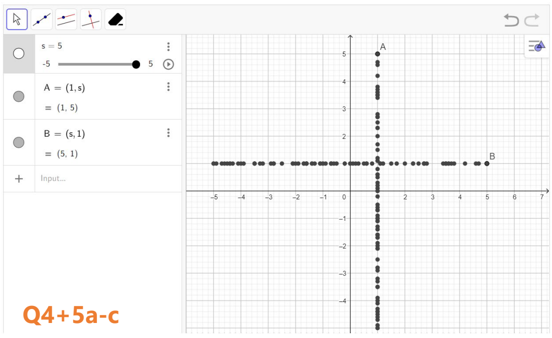
Answer guide:

In question 2 you have shown how you think A and B move.

You must argue why the points move exactly like that.

• Write what you know about the coordinates of the points.

• Write why this means that they must move exactly as you say.



Q4

Construct the points A = (1,s) and B = (s,1). It is important that you write *s* in the coordinate sets.

Q5

a) Change the value of s by dragging the slider.

b) Turn on “show trace” for the points (right click on the points)

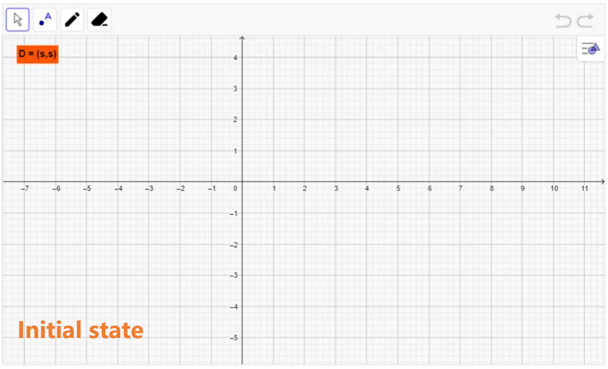
c) Change the value of *s* again by dragging the slider.

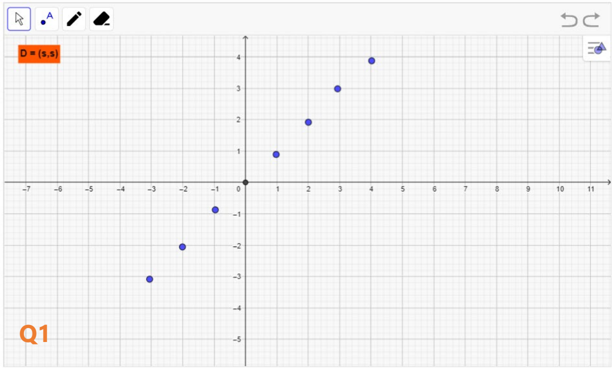
d) Explain to the camera how the points move.

e) Also explain why they move like that.

Figure 17 - Justified Prediction- Observation-Explanation task, iteration 2, problem set two

Titel: “Justified Prediction- Observation-Explanation” task of two-dimensional variable points





Q1

D = (s,s) and *s* is a variable.

Show in the coordinate system how point D moves when *s* changes value (You can use the tools in the toolbar, and you can also right-click).

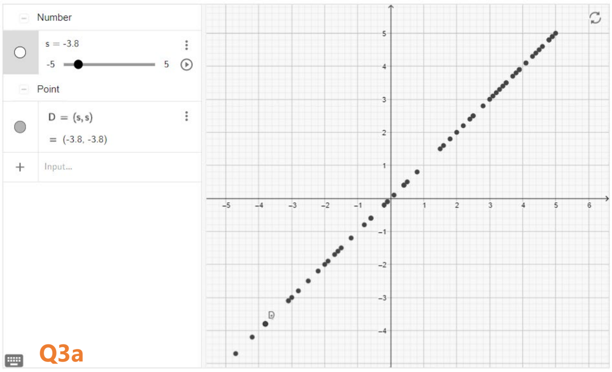
Q2

Justify your hypothesis - why does point D move as you claim?

Answer guide:

In question 1 you have shown how you think D moves. You must argue why D moves exactly like that.

Write what you know about D and why this means that D moves exactly as you say.



Q3

a) Type D = (s,s) into the GeoGebra app.

b) Change the value of *s* by dragging the slider.

c) Does the point move as you expected?

d) Describe here how the point moves.

e) Justify why D moves exactly like that.

Answer guide:

Consider your answer for question 2. Can you still use the argument after you have seen the point move in GeoGebra?

• If yes, copy it down here. Is there anything that needs to be added or changed?

• If no, formulate a new argument

Figure 18 - Justified Prediction- Observation-Explanation task, iteration 2, problem set three

Specifically, the restricted interface gives access to the 'move tool', 'the point tool', the 'trace function', and the 'pen tool'. The point tool enables the placement of free points on the coordinate plane, allowing subsequent movement using the move tool. When using the point tool, students must assign numerical values to each point they place. Multiple values of the variable can be depicted by plotting several points, shifting a single point, or activating the trace function, which leaves a track of points where the point is dragged across the screen. However, tracing can be challenging when moving a free object as it is susceptible to cursor movements. The point tool and trace function are designed for representing mathematical objects and properties. In contrast, the pen tool allows free drawing, requiring students to apply mathematical properties or functionality, such as the notation of values, sketching, plotting, tracing points, or drawing lines. Regarding RC, the prediction task required the students to expand their radius of action (Niss & Højgaard, 2019), as predicting variable points are a new kind of problem for them.

Furthermore, the restricted interface also required an unfamiliar use of the tools. Hence, the prediction tasks also involved processes of instrumental genesis. In paper 5, I demonstrate how the instrumental genesis process also contributes to the development of the technical aspect of RC, as the progressing complexity of tool use also necessitates inferences about the intrinsic properties of the prediction. However, the restricted interface also did cause confusion for some of the students. For instance, some would attempt to activate the algebra view or even open a new GeoGebra app to construct the points in the algebra view, obstructing the prediction step.

Paper 5, elaborate that the dynamic behavior of objects in a DGAE reflects the process-object nature of concept formation as either a discrete collection of examples or continuous movement. Miragliotta and Baccaglini-Frank (2021) describe that in predicting dynamic objects, students may pinpoint specific positions or envision, enact, or imitate continuous movements. This also holds true for variable points, as students can predict their dynamic behavior as shifting between positions in a coordinate plane or moving along a trajectory.

In paper 5, both discrete and continuous prediction is observed in the justification process of a pair of students, as they progress from predictions based on discrete examples to ones based on continuous movement. Naturally, both types are observed across the data set from these tasks, through the discrete is more common.

Recall that the nature of a prediction task requires that students draw on their own knowledge to make the prediction and justification. The observe-and-explain steps of the prediction task also allowed students to further elaborate, adjust, or advance the prediction and justification. Here follows an example from class C, school A.

In paper two, I present a short argument put forward by a pair of students concerning the first prediction task (see Figure 17). The students claim a structural relationship: that A and B form a slanted line. This is true but not a relevant answer to how the points move in the coordinate plane.

After the students test their prediction by observing the points move on the screen by animation, they explain that the points move more like a cross, though the points do form a slanted line. After observing the tracing of points, they also add that the points can continue tracing infinitely. In this case, the observe-and-explain step of the task, prompt the pair to adjust their prediction to focus on the trace of points and elaborate on more relevant properties.

The adjustment and elaboration of the prediction was very common in the observe-and-explain phase, particularly in the first of the prediction tasks. Properties that commonly appeared in the justification of predictions were equality, infinity, limits, that the variable has the same value in all terms it appears in, structural relationships of the point's trajectories with regard to the coordinate system, and the structural relationships of the line formed between A and B as having a constant slope.

For the second prediction task, the prediction was very short, and very few students justified it. I ascribe this to two factors. For one, in the second prediction task, the unfamiliar had become more familiar, which was evident as students were very confident in their prediction. Consequently, the students considered their prediction true and were less inclined to justify it. Secondly, the students became less engaged in the tasks as they tired near the end of the experiment.

The equal points task

The equal points task is discussed in papers 3 and 4. Paper 3 concerns iteration 2, and paper 4 concerns iterations 2 and 3. The task is part of the second set with one-dimensional variable points, and concerns points A and B from the first prediction task. For this task, the algebra view and its tools are accessible, but the toolbar is restricted to the cursor, the line construction tool, the parallel line construction tool, and the perpendicular line construction tool. The restriction on the toolbar was made to ensure that the students used the tools accessible in the algebra view. This task, particularly Q8, posed a significant opportunity for student's justification processes concerning the algebraic terms of the variable points, as related to equality, but it was missed by most students. In task A, all students constructed a point C that was not equal to point B for any value of *s*. In task B, the students were able to recognize that A and C could not be equal, but most students' justification relied on the geometric property of parallelism in the movement of the points.

Title: The “equal points” task”, iteration 2

Q7

a) When does A = B?

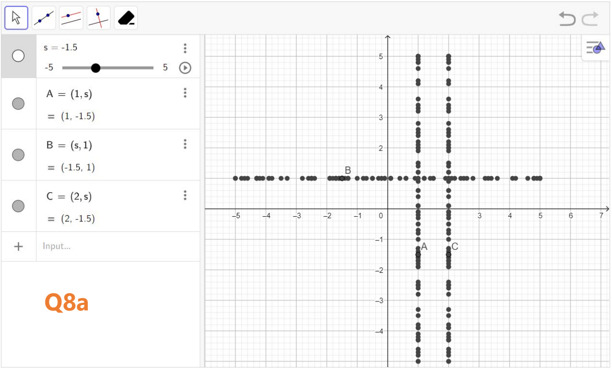
b) What arguments can you come up with to justify when A = B?

Answer guide:

For the answer you must see how many different arguments you can come up with, which justify that your answer in 7a) is correct.

Consider:

- What do you know about the points?
- What do you know about the variable s?
- What can you see?
- Why are the points not the same elsewhere?



Q8

a) Construct a new point C depending on s, which moves parallel to A (so *s* must be in the coordinate set of the new point).

b) Can C = A, and if so, when?

c) Can C = B, and if so, when?

d) Justify your answer.

Answer guide:

You must argue why the points are equal or not. Therefore, you must consider what it takes for them to be equal. Write what you find out as the first point in your argument.

- Write something about point B.
- Write something about point C.
- Write why this means that your answer must be correct.

If you find that C cannot be equal to B, try to see if you can change C so that they can. Maybe that can support your argument?

Figure 19 – The “equal points” task, iteration 2, problem set 2

In Q8c+d, all but one pair of students answered that C = B was not possible based on phenomenological justification. One pair of students did recognize that if they changed point C while maintaining parallelism in its trajectory to that of A, it would be possible for C to equal B. This pair was subjected to analysis in paper 3, while the whole data set of the task was analyzed in paper 4.

A significant discussion in a design context is why Q8 was inaccessible to most of the students.

Paper 3 indicated that the pair who did solve the task obtained a generalized view of variable points, as they considered point C one set of a collection of possible sets. In paper 4, I argued that this conception was a prerequisite for students’ engagement with the task.

Other valuable insights, in a design perspective, from paper 3 concern the phenomenological impressions of the variable points related to the algebraic terms in the coordinate sets, which students can experience and use in their justifications when solving the equal points task. One such insight was that the animation of a slider can give the impression that the speed of points depends on the coefficient. Another was that changing the coefficient could be experienced in terms of length of the trajectory of points and that the intersections of two trajectories indicate the coordinate position for when points are equal.

6.3.3 From HDH to design principles

The retrospective analysis of the two tasks can now provide insight allow for the addition of empirical arguments to the HDH, advancing them into design principles, proving concrete recommendations related to the normative elements of the design principles. The added insights are in boldface, and the design principles are in the form proposed by Van den Akker (1999) (see chapter 4.).

> If you want to design tasks for early secondary students to exercise their RC in justification about variable as a general number when using GeoGebra's algebra view and graphic view, you are advised that:

A. The *intrinsic* properties of the task must be known to the students but in problems novel to students (Duval, 2007; Niss & Højgaard, 2019), because it fosters a need for validation (Lithner, 2008). Intrinsic properties expressed in terms with variables may be signified using explicit variables rather than implicit ones. This is because implicit variables may cause students to focus on geometric properties rather that algebraic properties and that explicit variables provide students with access to direct manipulation of terms with a variable in the algebra view. **Students' knowledge about properties related to variables and algebraic terms, such as equality, infinity, and structural relationships, can be operationalized in justification processes through tasks about variable points. However, tasks that require students to have a generalized conception of variable points can prevent students from exercising RC. Furthermore, tasks where points move on parallel trajectories can deflect students to provide justifications of a geometric nature rather than algebraic.**

B. "Justified prediction-observation-explanation" tasks (White & Gunstone, 1992) can support students in forming claims and engage in justification processes about the dynamic behaviors of variable points and engage in justification about algebraic relationships and concepts based on their own knowledge (Kasmer & Kim, 2011; Lim et al., 2010) **of properties of the variable to justify patterned movements as algebraic relationships. The**

observe-and-explain phase urges students to elaborate and adjust their predictions according to observations. In addition, it is favorable to use a dynamic interface for the prediction rather than a paper environment. The dynamic interface allows students to represent variable relationships as variables, and dynamic tools allow students to predict variable points as both continuous and discrete movements. However, the restriction can be confusing, and students might disrupt the process by jumping ahead in the task, or by constructing points in the algebra view if made accessible.

C. Tasks where students create and manipulate symbolic representations that are simultaneously graphically represented can serve as a context for conjecturing about intrinsic algebraic properties related to variables (Kynigos et al., 2010). However, the complexity of the representational infrastructure of GeoGebra must be considered so that the representations students are to handle are accessible to them (paper 1). This might be obtained with variable points, as ordered pairs and simple algebraic expressions are familiar to lower secondary students. In addition, using notations of points depending on other points is not advisable for lower secondary students. **Using the animation feature of the slider can provide students with phenomenological impressions of speed as a dynamic property of a variable with a coefficient. If the trace function of a point is active, changing the coefficient in a variable point can be experienced in terms of the length of the trace, and on the intersections of traces, to indicate the coordinate position when points are equal.**

6.4 ITERATION 3: STRENGTHENING DESIGN PRINCIPLES

From the creative perspective, the following elaborates on the revision of tasks. Snapshots of the retrospective analysis are elaborated on, but contrary to the previous two iterations, the design principles are not revisited, as they are revised and discussed in the subsequent discussion.

6.4.1 The creative perspective

Several developments were implemented in the third iteration. The entire task sequences can be found in Appendix B. Due to the limitations of the kappa; I give a general description of the development of the sequence and only elaborate on the development of the prediction tasks and the equal points tasks.

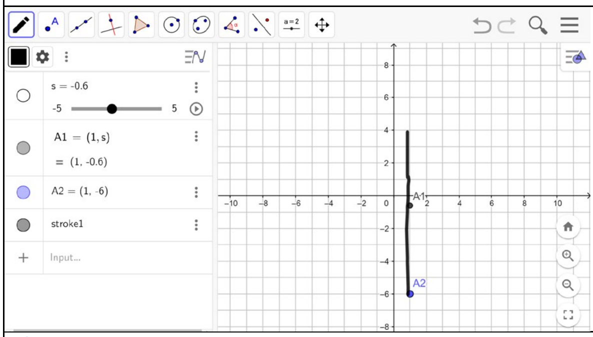
In the third iteration, the introductory tasks emphasized trace and animation and allowed students to explore variable points in a preconstructed GeoGebra worksheet. Entirely new tasks containing several sliders, were also introduced, to emphasize the difference between variable points that co-vary and points that do not. Some tasks with parallel moving points were redesigned so that the intrinsic properties focused on the coefficient rather than the geometric property of parallel trajectories, which deflected students from algebraic justification.

Developments concerning prediction tasks

To accommodate students' confusion with the restricted interface of the prediction tasks, the interface was changed to a regular GeoGebra interface. Instead, the task formulation described which tool to use for the prediction, and the first prediction task was introduced collectively in class. In addition, students were asked to specifically predict the trace.

The success of the prediction task led me to implement more prediction tasks with more complex terms than in iteration 2. In Figure 20, the prediction task that follow the students prediction of the point A1 = (1,s) (as in iteration 2) is presented. In this new prediction task, the students must predict the movement of point A2 = (1, *s*-1), requiring them to consider and justify how a constant term influences the position of the trace. This was possible for students to predict, as they had worked with traces in the microworld, which limits the variable to -5 and 5.

Title: "Justified prediction-observation-explanation" task of trace as related to constant terms



Q2

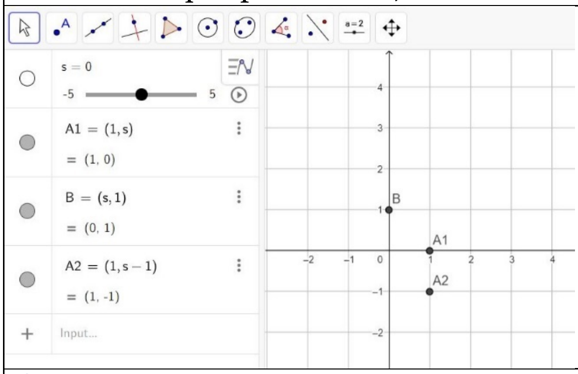
g) For this question you can use **the pen tool**, add **points,** and the **trace function**.
Guess and show how A2 = (1,*s* - 1) behaves in the coordinate system when changing the value of *s*.
h) Also show where the track will start and end.
i) Explain your guess.
j) Explain to each other how A2 is different from point A1.
k) Now construct point A2 = (1,*s*-1) and examine what happens when you change the value of *s*.
l) Does it match what you thought? Completely, partially, or not at all? Explain why.

Figure 20 – "Justified prediction-observation-explanation" task of trace as related to constant terms, iteration 3, problem set 2

Developments concerning the "equal points" task

Several adjustments were made to the equal points task. Some of them are also described and argued extensively for in paper 4. In the version of iteration 2, a point C was moving parallel to the trajectory of A. In the version of iteration 3, instead of C, point A2 moves on the same trajectory as A1. This change was made to accommodate two issues. For one, the parallel moving points deflected many students' attention from the terms in the algebra view, and thus, they gave a geometric justification for their answer (paper 4). Secondly, as the points now moved on the same trajectory, but in two different intervals, the task extended students' experience concerning the influence of constants on the position of the trace of variable points, from the prediction task presented in Figure 21.

Title: The "equal points" task, iteration 2



Q3 initial state

Q3

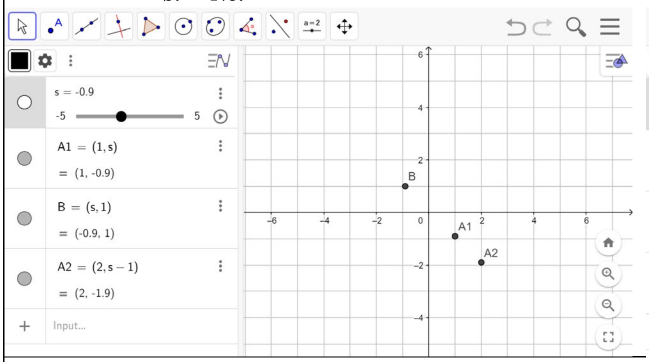
Note that: A1 = (1,s) and A2 = (1,s-1) and are constructed in the prediction tasks.
B has in question Q3 a-f) been defined as B = (s,1).

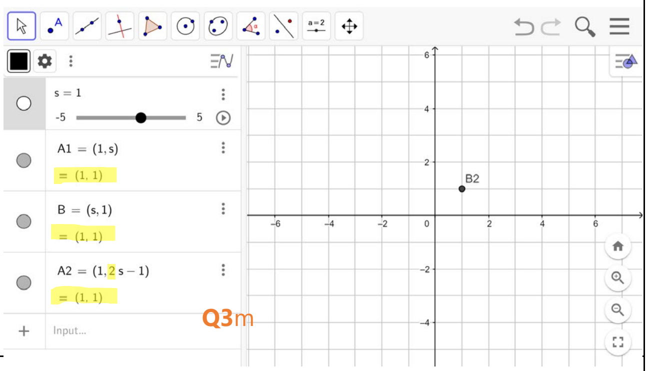
i) Can A1 = B? Please mark your answer and explain why/why not.

a. Yes when: ____
b. No.

j) Can A2 = B? Please mark your answer and explain why/why not.

a. Yes when: ____
b. No.





k) If you change A2's **x coordinate**, is it then possible for A2 = B?

a. Yes when: ____
Explain why your solution applies.
b. No. Explain why.

l) Change A2 back to A2 = (1,s - 1).
m) Now, if you change A2's **y coordinate**, can A2 = B without A2 = A1?

a. Yes when: ____
Explain why your solution applies.
b. No. Explain why not.

Figure 21 – The "equal points" task, iteration 3, problem set 2

The topmost concern of the equal points task was to make question 8c–d (see Figure 19) more accessible to students. This was addressed by formulating the questions more concretely, with clear instructions. The formulation now simply directed students to change the coordinates of A2 to obtain equality to B by transforming the x and y coordinates (see question Q3, k–m in Figure 22).

6.4.2 Snapshots from the retrospective analysis

The students from school B, participating in the third iteration, had very little experience with using GeoGebra. Consequently, many students experienced technical issues with the algebra view. 7 pairs out of 18 did not answer the task due to technical issues or misunderstanding the task. Moreover, it also became apparent that students who struggled with identifying intrinsic properties in the geometric view was further challenged by the information in the algebra view. An example of such issues is demonstrated by a pair of students who put forward a faulty justification for question Q3 k): If you change A2's x coordinate, is it then possible for A2 = B?

Yes, when A2 = (2, s-1) because A2 and A1 are then on different trajectories.

The students focus on the change of trajectory of A2 as a result of changing the y coordinate from 1 to 2 using the algebra view. They seem focused on the result of manipulation rather than the intrinsic properties of the task. The students fail to relate the justification to equality between the relevant points *B* and *A2*. Therefore, even though the students do refer to the coordinate set, they struggle to identify intrinsic properties in their justification.

Thus, it is advisable that the students have some experience with GeoGebra's graphic view prior to working with variable points tasks. I address this further in relation to the equal points task.

In general, less students engaged in justification in iteration 3 compared to iteration 2. This is evident in paper 4. In iteration 2, all students who provided a solution also justified it (between 10 and 13 pairs of students out of 17). In iteration 3, however, only between 4–6 pairs of students justified their answer out of the 10 pairs who provided a solution. This reflects the general picture of the two iterations.

This can be explained by several factors, e.g., that students had been participating in online schooling for long periods of time, giving them less exercise with RC. In general, school B put less emphasis on mathematical competencies than school A, and their regular teacher did not attend on the day of the experiment, due to covid, so the students were less prepared and supported during the experiment, as the substitute teacher was not informed about it.

The new prediction task

The new prediction task (see Figure 20) of the point A2 = (1, *s*-1) was more challenging than the prediction tasks from iteration 2. Several pairs made faulty predictions and justifications. For instance, several pairs predicted that the point moved on x=-1 or that the point would move slower when animated. This made the observe-and-explain step of the task even more significant for the justification process, as students needed to adjust their faulty prediction.

The change from a restricted interface to a regular interface, with the algebra view active, meant that students could test their prediction on top of what they had drawn with the pen, making details of their predictions more explicit in the test-and-observe step. For example, by showing how the drawn length or placement of the trace corresponded with the test. In addition, students could more easily compare new predictions to those already tested. Question D, requiring students to compare A1 and A2, directed students' justification to explain how the constant term influenced the trace and the comparison of variable points in the prediction, ensuring that students put forward justifications that related the trace to the algebraic terms in the ordered pairs.

Unfortunately, there were still students who were confused by the prediction tasks and would jump ahead to constructing and testing the points, so that problem was not eliminated.

The "equal points" task

The new task design and the more direct formulation of questions (c–d of iteration 2; k, l, m of iteration 3) did overcome the identified issue of the equal points task. In the third iteration, there was no geometric justification, and there was an increased number of students who attempted to solve question M.

Students quite easily found a solution for question K, obtaining equality between A2 and B by changing the constant in the y coordinate, and there were no geometric arguments. Instead, there was a greater diversity among arguments of those who did justify their answer. Students' answers also provided additional insights into the phenomenological impressions students can have in the task. In paper 4, I discuss the students' phenomenological impressions of trace and intersection in their justification of solutions to the equal point tasks, in which some students use the phenomenological impression to argue for structural relationships. For example, consider these justifications:

> Can A2 = B? (A2 = (1, *s*-1), B = (*s*,1))
>
> *No, as there will always be one point that has a distance to the intersection when the other one is at the intersection.*

And

If you change A2's **x coordinate**, is it then possible for A2 = B?

Yes, when A2 = (2, s-1), because then A2 and B have the same distance to the intersection of the trajectories.

These justifications are phenomenological in nature but also imply that the students understand the relationship between changing the x coordinate and the position of the trajectory of the point left by the trace. Moreover, they understand that the intersection of these trajectories indicates possible equality between the points.

In paper 4, I also relate such justifications to students' epistemic mediation in their use of the graphic view, and I argue that it can be a steppingstone for students' epistemic mediation in their use of the algebra view, if they are challenged to relate their justification to the ordered pair of the variable points.

Concerning question M, "if you change A2's y coordinate, can A2 = B without A2 = A1?", there was an increased number of students who attempted to solve question compared to question C in iteration 2. This shows that the direct formulation of the question made it more assessable and engageable for both problem-solving and justification. Still, many students did not obtain equality between point A2 and B. In paper 4, I relate this to the problem handling competency and the symbol and formalism competency.

The ability of students to make use of the algebra view as a tool for justification is closely linked to their understanding of symbolism and formalism. Incorrect answers may result from incomplete knowledge of algebraic rules and procedures. Furthermore, if the student does not attempt to justify their solution, their mistakes may go unnoticed.

An issue that relates to student problem handling competency is that some students rely too heavily on a single strategy. Four pairs out of the ten could justify why their technique did not provide a sound solution, yet they were reluctant to attempt another technique. Consequently, the lack of a solution becomes an argument, and hinders the students exercise of RC that has creative qualities.

6.5 DISCUSSION, PART 1: DESIGN PRINCIPLES AND DESIGN

In this section, I discuss the results in answering the research question of the chapter:

In what ways can tasks be designed to encourage lower secondary students to exercise their RC when using a DGAE in the case of justification focusing on variables as a general number?

I will articulate and discuss the answer in the form of three particular but related results of the design processes. The first results are the design principles also considering contributions of iterations 3. The second result is the microworld of *variable points*, and the third result are the concrete tasks presented in the chapter.

6.5.1 Design principles

The design principles are the main contribution toward answering *in what ways*. As is the nature of design research, the design principles are particular to the design. However, some aspects can be considered from a more general perspective of reasoning in mathematics education, which I will debate after presenting the principles.

The principles have descriptive theory elements (Prediger, 2019) that describe features of tasks and phenomena, such as particular phenomenological impressions that can occur. The design principles have predictive theory elements (Prediger, 2019) that argue for certain solutions and actions toward a given aim or problem, or predict outcomes of actions, design elements, or structural elements. Explanatory theory elements of certain phenomena are closely related to the theoretical development (Prediger, 2019), which are presented in the subsequent chapters. Hence, explanatory elements are discussed in the final discussion in Chapter 9.

In the following, I present the final design principles A, B and C. Again, they are presented in the form proposed by Van den Akker (1999) (see chapter 4). New contributions from iteration 3 are added in bold.

> If you want to design tasks for early secondary students to exercise their RC in justification about variable as a general number when using GeoGebra's algebra view and graphic view, you are advised that:

A. The intrinsic properties of the task must be known to the students but in problems novel to students (Duval, 2007; Niss & Højgaard, 2019), because it fosters a need for validation (Lithner, 2008). Intrinsic properties expressed in terms with variables may be signified using explicit variables rather than implicit ones. This is because implicit variables may cause students to focus on geometric properties rather that algebraic properties and that explicit variables provide students with access to direct manipulation of terms with a variable in the algebra view. **In order for students to be able to recognize the intrinsic properties of a task, it is recommendable that students have experience with using the graphic view for epistemic mediation, prior to working with variable points.** Students' knowledge about properties related to variables and algebraic terms, such as equality, infinity, and structural relationships can be operationalized in justification processes through tasks about variable points. However, tasks that require student to have a generalized conception of variable points can prevent students (at a secondary level) from exercising RC, **which can be addressed by direct action-oriented formulations of question instructing students to modify terms in the algebraic expressions. Finally, despite** parallelism being a known structural relationship for the students, tasks with parallel relationships can deflect students to provide justification of a geometric nature rather than one related to the variable.

B. "Justified prediction-observation-explanation" tasks (White & Gunstone, 1992) can support students in forming claims and engage in justification processes about the dynamic behaviors of variable points and engage in justification processes about algebraic relationships based on their own knowledge (Kasmer & Kim, 2011; Lim et al., 2010) of properties of the variable to justify patterned movements as algebraic relationships. **Predictions of variable points containing a constant term along with the variable in the ordered pairs can provide students with the opportunity to predict and justify the placement of traces as related to the constant. Comparison of two or more points in the prediction can engage students to justify the positions of traces as related to the differences in algebraic terms.** The observe-and-explain phase urges students to elaborate and adjust their predictions according to observations. In addition, it is favorable to use a dynamic interface for the prediction rather than a paper environment (Noss et al., 2012), because the dynamic interface allows students to represent variable relationships as variable and to predict variable points as both continuous and discrete movement. **Predictions of variable points in the GeoGebra environment also allow students to test constructed points, on top of their prediction, making small differences between predictions and tests stand out visually. Expect that students need close guidance in prediction tasks as the prediction step can confuse students** and students might "disrupt" the process by jumping ahead in the task and constructing points in the algebra view.

C. Tasks where students create and manipulate symbolic representations that are simultaneously graphically represented can serve as a context for conjecturing about intrinsic algebraic properties related to variables. However, the complexity of the representational infrastructure of GeoGebra must be considered so that the representations students create and manipulate are familiar to them (based on review). This might be obtained with variable points, as ordered pairs and simple algebraic expressions are familiar to lower secondary students. In addition, using notations of points depending on other points is not advisable for lower secondary students. Using the animation feature of the slider can provide students with phenomenological impressions of speed as a dynamic property of a variable with a coefficient. If the trace function of a point is active, changing the coefficient in a variable point can be experienced in terms of the length of the trace and on the intersections of traces to indicate the coordinate position when points are equal. **Students can also experience equality between variable points, as dependent on the distance to an identified intersection of traces. Students' identification of structural relationships through the phenomenological impression obtained from the geometric view, and through epistemic mediation, can be considered a stepping stone toward justifying those relationships in algebraic terms. However, students' exercise of RC as justification also depends on their problem handling competency and their symbol and formalism competency.**

In the research question, I ask *in what ways,* emphasizing that the design and design process has materialized in certain ways and not in other possible ways. The following highlights choices taken in the design process and discus how these choices have addressed issues concerning RC and DGAE.

Students often face challenges with a high level of complexity in both reasoning and the use of DGAE. In reasoning, this complexity involves understanding the relevant concepts and properties of a problem (Duval, 2007). In the DGAE, it consists in understanding the representational structures in which they are expressed. Principles A and C have the same normative element of familiarity as an approach to overcome the high level of complexity. As a result, they allow students to focus on justification processes, but with elements of novelty to progress a need for justification. This is *a way* to approach design that emphasizes the exercise of *competency* rather than conceptual development, in which learning of unfamiliar concepts are the goal.

In the theoretical foundation (Chapter 2) and in paper 5, I describe Vergnaud's (1998b) concepts of schemes. Particularly in paper 5, I discuss how we may see students' justification processes in the light of possibilities of inference between theorems-in-action about concepts-in-action. In this perspective, familiarity means that students have some theorems-in-action about the intrinsic properties but must create inferences between theorems-in-action that are particular to the task they are solving. Hence, through the perspective of Vergnaud's schemes (1998b), the normative theory element of familiar concepts and properties in novel problems create possibilities for inference.

There are two central challenges in design toward such goal: what properties are familiar to the target group, and which problems are novel. Concerning the first, the diversity of students' knowledge is a fundamental precondition, so despite the approach of familiar concepts and properties, descriptive theory elements are needed to describe what properties are suitable for the target students. The generalized conception of variable points is an example of an overly complex property, excluding students' engagement in justification, whereas infinity, limits, and equality are concepts used in justification by most students in the experiments.

The second challenge of novel tasks has been obtained through the microworld. The microworld itself has a familiar and novel aspect for the students. As argued in the analytical perspective, the ordered pairs are indeed familiar to students, but this was not the case for the variable and algebraic terms in this context. This has allowed for the creation of tasks unlike regular mathematical tasks, which are novel to all students.

Design principle B concerns a particular kind of task structure as a way of facilitating justification processes, and hence differ from A and C. This reflects a choice in the focus of the design chapter. There are other kinds of tasks that I could have emphasized and explored in the principles, e.g., construction tasks or pattern generalization tasks. The choice to emphasize "justified prediction-observation-explanation" as *a way* to design reflects a personal curiosity of how such tasks might be designed in the context of DGAE. More importantly, the task has a strong relevance to the practice community in promoting justification in the classroom through a task structure that is adaptable to different subjects and environments. Principle B is hence an elaboration of how A and C can be operationalized into a task structure.

6.5.2 The microworld

We create microworlds with the hope that it makes the abstract world of mathematical concepts easier for students to grasp. The microworld of variable points is indeed developed with such intent. It builds on the tradition of microworlds to develop visual representations to explore algebraic expressions, but contrary to many others (e.g., MiGen and MoPiX), it exploits a regular DGAE accessible to students, and it is embedded in the representational structure of coordinate geometry. Though the microworld of variable points is designed for students' exercise of RC, *variable points* hold potential for the exercise of other competencies and learning of basic algebraic concepts. In the progression of coordinate geometry, there is a big gap in comprehension and algorithms, from placing ordered pairs as points in the coordinate system to the next theoretical step, the distance formula or linear function in linear algebra. The microworld of variable points has potential to lessen that gap, as it is an explorative environment where students can progress their understanding of algebraic concepts, variables, and terms in the context of the coordinate system.

6.5.3 Tasks for the exercise of RC in justification

I have presented two kinds of tasks that have proven to be particularly valuable in the context of microworld for students to exercise RC in justification: the "justified prediction-observation-explanation" tasks and the "equal points" task. However, these two task types have very different prerequisites. The first task requires *translation* between representations in the microworld, while the equal points task requires *transformation* of the symbolic representation to reach an equality requirement between two points. Also, they both represent different ways to actuate the dynamic properties of variable points.

The presented tasks are examples of how such tasks can be formulated. However, both can be redesigned to present more or less complex problems and focus on different properties. As such, I would like to explicate them as *types* of tasks that invite students to exercise their RC, and as particular results of the study.

7 RELATING SCHEME-TECHNIQUE AND REASONING COMPETENCY

In this chapter, I address RQ2:

> What are the relationships between lower secondary students' scheme-technique duality when solving tasks developed for RQ1 in a dynamic geometry and algebra environment and their exercise of reasoning competency as justification?

Recall that when evaluating RC, three dimensions are taken into account: degree of coverage, radius of action, and technical level (Niss & Højgaard, 2011, 2019) (see also 0). Coverage pertains to a competency's various aspects, such as actively participating in different forms of reasoning. The radius of action encompasses the diverse contexts in which the competency can be applied, spanning various domains and social situations. Lastly, the technical dimension addresses the sophistication of concepts, theories, and methods. In addressing RQ2, the relationships between students' scheme-technique duality and these three dimensions of RC are drawn out by identifying differences between student pairs and the progression of instrumental genesis for an individual student. Relevant results from papers 3, 4, and 5, as well as additional results, will be presented to draw out these relationships. Consequently, the wording in RQ2: "for tasks developed for RQ1" can be concretized to the prediction tasks and the "equal points" task presented in chapter 6. Furthermore, as these papers examine the scheme-technique duality (Drijvers et al., 2013) through the notion of instrumented justification (IJ), IJ is elaborated in the forthcoming section 7.1 before presenting relevant results.

7.2 delve into results concerning the "equal points" task, and 7.3 into results concerning the "Justified prediction-observation-explanation". Each draws on results from papers but also presents additional analysis and results. 7.4 is the second part of the discussion, answering RQ2.

7.1 THE SCHEME - TECHNIQUE DUALITY AS INSTRUMENTED JUSTIFICATION

IAME is usually applied to analyze students solving a type of task. For instance, in the case analyzed by Drijvers et al. (2013), a student solved quadratic equations using the CAS tool. Thus, typical for cases analyzed in IAME, the main goal in students' schemes is to solve a problem, with associated subgoals particular to the process. However, the processes of justification differ as the main goal is to change the epistemic value of a mathematical claim that concerns the solution to a problem or strategy. Hence, the scheme-technique duality in justification must be reconsidered toward such a goal. In Toulmin's argumentation model (Toulmin, 2003), the goal can be considered related to the element "qualifier", which indicates the perceived probability of the claim. This is the basis for reinterpreting the scheme-technique duality using Toulmin's argumentation model in the creation of an analytical tool (see Figure 22) for IJ processes. In MER, Toulmin's model is usually used to analyze a finished argument or chains of sub-arguments, while the scheme-technique duality is expressed in processes. Toulmin's model is hence adapted to capture the process from forming a claim to restating that claim, along with a change in the qualifier when using an artifact. This process involves changing the qualifier from 'possible' to 'more possible', 'less possible', 'true', or 'false'. The change is obtained by generating data through techniques as evidence to support or refute the initial claim.

The analytical tool highlights this close relationship between data and techniques, correlating a technique to the data it produces through connected frames. The schemes (Vergnaud, 1998b) (see also section 2.5) that direct and organize techniques contain conceptual elements and rules that regulate actions seen as warrants connecting the data to the claim. These warrants can be inferred from students' techniques and verbal expressions (Rezat, 2021).

Figure 22 shows a generic diagram of the IJ analytical tool as an adaption of Toulmin's model. In continuous sub-processes, the first uttered claim, along with its qualifier, is noted in the top right corner in grey, and below is the re-claim with a new qualifier. Finally, the rebuttal consists of the limitations of the claim, or counterarguments, as in Toulmin's (2003) original model.

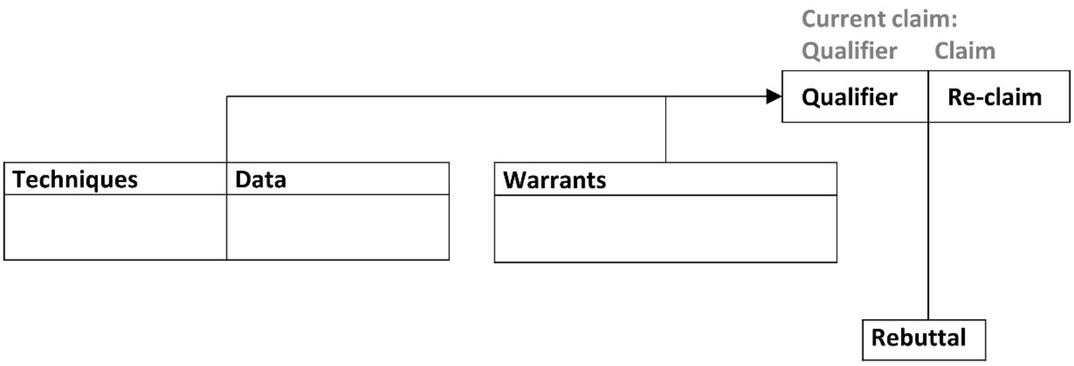


Figure 22 – Adaptation of Toulmin's model into an analytical tool for students' instrumented justification

Based on the analytical tool, IJ is described as: "*a process through which a student modifies the qualifier of one (or more related) claim(s) using techniques in a digital environment to generate and search for data and warrants constituting evidence for such claim(s)*" (Paper 3, p. 135; italics in original).

It is important to stress that IJ complies with the theoretical notions of the IAME, which are still intact. For instance, epistemic and pragmatic mediations are still considered a part of such processes and students' actions are emphasized as oriented toward goals and subgoals, conforming with the notion of scheme.

Paper 4 and paper 3+5 have different perspectives on IJ. Paper 4 explores the nature of students' final arguments in the complete set of data from iteration 2 and 3 of the equal points task (see section 5.3). The overview of students' arguments is then used in an analysis and discussion of illustrative cases, concerning the nature of arguments and students' mediations and goals as related to their RC. In paper 3 and 5, I investigate singular cases of IJ for in-depth analysis of the processes. Hence, the papers provide different types of results to identify relationships between students' scheme-technique duality in IJ processes and RC.

7.2 MEDIATIONS AND TECHNIQUES EMERGING FROM THE EQUAL POINTS TASK

In the following section, perspectives of paper 4 are addressed. First, some results from paper 4 and additional data are presented. Then through analyzed I draw connections between students' mediations in the IJ processes, their goals and techniques to the three dimensions of RC.

7.2.1 Results concerning student's justifying solutions

The first set of results stems from paper 4. Table 3, showing students' answers to the equal points task is found below. The answers are grouped according to the final argument (question numbers are correlated with how they are presented in subsection 6.3.2). The phrasing of the arguments are presented in a condensed form, which allows grouping similar arguments. The grouping provides an overview of the number of student pairs that engaged in justification processes about their solutions and the nature of their final arguments. It appears from Table 3 that only some students justified their solutions. It is not evident, however, how different pairs of students progressed throughout the task, which could provide a more comprehensive understanding of how they exercised their reasoning.

Table 3 – Grouped student answers for the equal points task from third iteration and the nature of their justification referring to either A) an algebraic relationship, P) phenomenological impressions, or N) numeric information

Grouped student answers, n = 18 pairs	n pair(s)	Nature: A, N, P
i) Can $A1 = B$ and when? *(Yes, when $s = 1$)*. Justify your answer.		
Yes, when $s = 1$, as $B = (s,1)$ and $A1 = (1,s)$, and when s is one, they are both (1,1)	2	A
Yes, when $s = 1$ because then both points are (1,1)	1	A
Yes, when both coordinate sets are (1,1), no justification	3	-
Yes, as the points cross each other	1	P
Yes, when $s = 1$, no justification	3	-
Irrelevant answer	7	-
No answer	1	-
j) Can $A2 = B$ and when? *(No)*. Justify your answer.		
No, as $A2$ is always one below $A1$ because of "the -1" (and $B = A1$)	2	A
No, as there will always be one point that has a distance to the intersection when the other one is at the intersection	3	P
No, they are never at the same place at the same time	1	P
No, no justification	4	-
Irrelevant answer	7	-
No answer	1	-
k) If you change the x-coordinate of $A2$, is it then possible for $A2 = B$? • If yes, when and why? *(Yes, more solutions, e.g., if $A2 = (2,s-1)$ and $s = 2$)* • If no, why not?		
Yes, when $A2 = (2,s\text{-}1)$, because then both points can have the x-value of 2 when s is 2	2	A
Yes, when $A2 = (2,s\text{-}1)$, because then $A2$ and B have the same distance to the intersection of the trajectories	1	P
Yes, when $A2 = (2,s\text{-}1)$, because $A2$ and $A1$ are then on different trajectories	1	P
Yes, when $A2 = (2,s\text{-}1)$, no justification	6	-
Irrelevant answer	7	-
No answer	1	-
l) If instead, you change the y-coordinate of $A2$, is it then possible for $A2 = B$ without $A2 = A1$? • If yes, when and why? *(Yes, e.g., $A2 = (1,2s-1)$ or $A2 = (1, -s-1)$)* • If no, why not?		
Yes, when $A2 = (1,2s\text{-}1)$, no justification	1	-
No, because B always has a distance to the intersection of B and $A2$'s trajectories when $A2$ is at the intersection	2	P
No, because they are equal in (1,1) where $A1 = B$, and we cannot find a solution where $A2$ does not equal $A1$	4	A
Yes, when $A2 = (1,s1)$ or $A2 = (1,s)$, no justification	2	-
No, no justification	1	-
Irrelevant answer	7	-
No answer	1	-

Table 4 presents additional data for students who justified their solutions to at least one question. Each pair's answers and justifications are color-coded as follows: Correct answer, partly correct answer, wrong answer, phenomenological based justification, algebraically based justification.

By considering justification of an algebraic nature as more complex than those of a phenomenological nature, the student's exercise of RC in the task, across all questions in the equal points task, can be qualified. In the table, the pairs of students are arranged according to their exercise of RC. Pair 1, to the far left, exercises the least developed RC, while Pair 7, to the far right, exercises the most developed RC.

Table 4 - The answers and arguments of justifying students in answering the equal points task iteration 3. Coding: Correct answer, partly correct answer, wrong answer, phenomenological based justification, algebraically based justification

Pair # Question/ answer	Pair 1	Pair 2	Pair 3	Pair 4	Pair 5	Pair 6	Pair 7
i) Can A1 = B and when?	Yes, when *s=1*	Yes, when both coordinate sets are (1,1)	Yes, when *s=1*	Yes	Yes, when *s=1*	Yes, when *s=1*	Yes, when *s=1*
Justification	not justified	not justified	not justified	As the points cross each other	because then both points are (1,1)	As B = (s,1) and A1 = (1,s), and when s is one, they both are (1,1)	As B=(s,1) and A1 = (1,s), and when s is one, they both are (1,1)
j) Can *A2* = *B* and when?	No	No	No	No	No	No	No
Justification	There will always be one point that has a distance to the intersection when the other one is at the intersection	not justified	There will always be one point that has a distance to the intersection when the other one is at the intersection	No, they are never at the same place at the same time	A2 is always one below A1 because of the -1" and B = A1	There will always be one point that has a distance to the intersection when the other one is at the intersection	A2 is always one below A1 because of "the -1"
k)Can A2 = B and when? x-coordinate	Yes, when A2= (2,s-1)	Yes, when A2=(2,s-1)	Yes, when A2=(2,s-1)	Yes, when A2=(2,s-1)	Yes, when A2=(2,s-1)	Yes, when A2=(2,s-1)	Yes, when A2=(2,s-1)
Justification	not justified	not justified	Because A2 and A1 are then on different trajectories	not justified	Because then A2 and B have the same distance to the intersection of the trajectories	Because then both points can have the x-value of 2 when *s* is 2	Because then both points can have the x-value of 2 when *s* is 2
l) Can A2=B and when? y-coordinate	Yes	No	No	No	No	No	No
Justification	not justified	Because they are equal in (1,1) where A1 = B, and we cannot find a solution where A2 does not equal A1	Because B always has a distance to the intersection of B and A2's trajectories when A2 is at the intersection	Because they are equal in (1,1) where A1 = B, and we cannot find a solution where A2 does not equal A1	Because they are equal in (1,1) where A1 = B, and we cannot find a solution where A2 does not equal A1	Because they are equal in (1,1) where A1 = B, and we cannot find a solution where A2 does not equal A1	Because B always has a distance to the intersection of B and A2's trajectories when A2 is at the intersection

7.2.2 Analysis of goals, mediation and techniques and their relationships to RC

The analysis points to differences between the seven pairs, which is then related to one of the three dimensions of the RC to establish relationships between students' IJ processes and the RC.

Based on Tables 3 and 4, the following three analyses explore relationships between students' goals in the mediation of artifacts in IJ processes and their RC.

Goal of tool use as related to degree of coverage

The close relationship between RC and problem handling in justification (see section 2.3) is significant in the following analysis.

In the justification of a solution, but also during the problem-solving, students enter IJ processes. Thus, in IJ there is an interplay of both problem handling competency and RC. Delineating which processes relate to which is significant to provide an answer to the RQ. Drawing on the notion of schemes that organize *goal*-oriented activities (Vergnaud, 1998b), the two competencies can be delineated with respect to the goal of the student's activity, as either reaching a solution to a given problem or changing the epistemic value of a claim stated during the problem-solving process or as a solution. Some of the students engage in pragmatic and epistemic mediation toward the goal of problem solving, but rarely or never toward justifying their solution. Hence, they provide no, or only very few, justifications for their solutions, e.g., Pairs 1 and 2 in Table 4. These two pairs are students who can solve most questions but not necessarily justify their answer. They might exercise justification during their problem solving, but we cannot conclude that they do based on the results. However, we can say that the students who justify their solution display a more developed degree of coverage than students who do not.

As the degree of coverage is the aspect of RC that students exercise, we can consider students' degree of coverage in relation to their goal of using an artifact. The goal must be aimed at changing the epistemic value of a claim within a problem-solving context. It follows that especially in justification processes, students must shift their goal from problem-solving towards justification.

Pragmatic and epistemic mediation as related to radius of action

This analysis identifies differences in students' mediations of artifacts in the algebra view and the graphic view, which is then related to the radius of action of RC.

In order for students to understand how their techniques influence the dynamic behavior of variable points, they must comprehend the how the representation of the objects in the graphic view relate to the variable points in the algebra view. This comprehension requires epistemic mediation of both the graphic view and the algebra view. In general, for students to put forward arguments that relate algebraic properties to the representation in the graphic view, they must be able to use pragmatic and epistemic mediation of both the graphic view and the algebra view toward the goal of justification. The results in Tables 3 and 4 show that only a small number of students connect the phenomenon they observe in the graphic view to the coordinate set and the algebraic expressions in the algebra view regarding their solution. Pairs 6 and 7 do so most consistently of the seven pairs.

The students who primarily present phenomenologically natured arguments (Pairs 3 and 4) mostly attain epistemic mediation of the graphic view. The input field and the slider tool in the algebra view are only used to produce data, so the students only attain pragmatic use of the algebra view. This is further nuanced by two cases of phenomenologically natured arguments, also presented in paper 4. They differ by the fact that in one case, students justify with regard to the intrinsic properties (Lithner, 2008) of the task, where the students in the other case do not. Consider the differences in the following two examples, both in the context of question K.

Pair 5: "Yes, A2 = (2,s-1), because then A2 and B have the same distance to the intersection of the trajectories."

Pair 3: 'Yes, when A2 = (2, s-1), because then A2 and A1 are on different trajectories.

The argument of Pair 5 considers the intrinsic properties, such as equality of A2 and B at their intersection of trajectories, and indirectly relates *s-1* to A2's distance to that intersection.

Pair 5's justification indicates that students understand that the constant in the coordinate set corresponds to the position of the trajectory of the point, but they fail to relate it to equality, in general and between the relevant points B and A2. As argued in paper 4:

> The next step is for the students to instrumentalize components of the algebra view for justification. Or in other words, they must evolve their use of the algebra view to also encompass epistemic mediation for the goal of justifying their answer. In that sense, some of the justifications that rely on phenomenological experiences can be a steppingstone for students exercising their RC in the algebraic domain (p.30).

On the contrary the argument of Pair 3, though also phenomenological in nature, does not relate to intrinsic properties of the task. Thus, even though the students do refer to the coordinate set, these students struggle to identify the core concepts of the problem and exercise a less developed RC.

Finally, Pairs 1 and 2 mostly did not justify their solutions. It is, however, possible that they have a pragmatic mediation toward justification. If considering their solution as evidence, they are performing verification, which is a pragmatic use of the algebra view and graphic view toward the goal of justifying (elaborated in following section).

By this analysis, we can draw out differences in students' mediations of the graphic view and algebra view.

- No epistemic mediation toward the goal of justification. Possibly, there is pragmatic use of the algebra view and graphic view toward the goal of justification, such as verification.
- Pragmatic mediation in the use of artifacts in the algebra view, but epistemic mediation of the graphic view toward the goal of justification. This is further nuanced by the mediation regarding intrinsic properties or not.
- Both pragmatic and epistemic mediation of the algebra view and graphic view.

The radius of action relates to the situations and domains students exercise the competency in. Hence, delineating IJ during problem-solving and justification of the solution is related to radius of action. Considering the different views of GeoGebra as different contexts, i.e., the graphic view and the algebra view each representing different mathematical domains, students' pragmatic and epistemic use of specific artifacts in different views relates to the radius of action of their RC.

Finally, there are nuances of students' epistemic mediations. In IJ processes, pragmatic mediations are the production of data through techniques, and epistemic mediation is the interpretation of data through warrants. The quality of the epistemic mediation depends on whether the mediated involves intrinsic properties. This is, however, related to students' technical level, which I analyze in the following section.

7.2.3 Students' techniques in the exercise of RC

Up until now, the analysis of student activities and their goals, along with their mediations, has been conducted on the basis of Tables 3 and 4, and in relation to the degree of coverage and radius of action. The third dimension, which is the technical level, requires data on the specific techniques used. To maintain the accuracy and relevance of the analysis, the results related to this dimension are presented in a separate section. Further analysis will be provided after this presentation.

Additional findings

Describing students' techniques of both the input field and slider tool provides further evidence of how access to construction and manipulation of algebraic expressions in conjunction with the slider tool relates to students' RC. I focus particularly on question K and L of the equal points task, as they involve techniques in the input field of the algebra view in combination with the slider. Table 4 presents sequences of techniques, considered as the generative aspect of schemes (Vergnaud, 1998b), across the seven pairs.

For question K, "Can A2 = B if you edit the x-coordinate of A2?", students must change the numeric value to either 2 or *s*. By scrutinizing which techniques students employ to reach the solution, and in some cases a justification, the following sequences appear:

$T_{E,V,J}$: Edit the value of x-coordinate to 2, drag slider for verification, drag slider for IJ (n=4)

$T_{E,V}$: Edit value of x-coordinate to 2, drag slider for verification (n=2)

T_E: Imagine editing the value of x-coordinate to 2 (n=1)

In Table 5, the sequences of techniques are put in relation to the student pairs in Table 4.

Table 5 – Techniques of 7 pairs of students answering question k). Coding: Correct answer, partly correct answer, wrong answer, phenomenological based justification, algebraically based justification

Pair #	Pair 1	Pair 2	Pair 3	Pair 4	Pair 5	Pair 6	Pair 7
Can A2=B? x-coordinate	Yes, when A2= (2,s-1)	Yes, when A2=(2,s-1)	Yes, when A2=(2,s-1)	Yes, when A2=(2,s-1)	Yes, when A2=(2,s-1)	Yes, when A2=(2,s-1)	Yes, when A2=(2,s-1)
Technique	T_{IE}	$T_{E,V}$	$T_{E,V,J}$	$T_{E,V}$	$T_{E,V,J}$	$T_{E,V,J}$	$T_{E,V,J}$
Justifi-cation	not justified	not justified	Because then A2 and A1 are on different trajectories	not justified	Because then A2 and B have the same distance to the intersection of the trajectories	Because then both points can have the x-value of 2 when *s* is 2	Because then both points can have the x-value of 2 when *s* is 2

For question L, “Can A2 = B if you change the y-coordinate of A2?”, a successful technique is to multiply with a coefficient of 2 without deleting the term. By scrutinizing which techniques the seven pairs of students in Table 4 employ to reach the solution, and in some cases a justification, for question L, the following sequences appear:

$T_{T,V,J}$: Repeatedly edit the term at random and drag slider for verification. Drag slider for IJ (n=5)

$T_{-T,c\rightarrow 1,V,J}$: Delete the term and repeatedly edit the coefficient by approaching one, drag slider for verification, drag slider for IJ (n=1)

$T_{-T,C\rightarrow 1,V}$: Delete the term and repeatedly edit the coefficient by approaching one, drag slider for verification (n=1)

$T_{C,V}$: Keep the term and edit the coefficient, drag slider for verification (n=1)

In Table 6, the sequences of techniques are put in relation to the student pairs in Table 4.

Table 6 - Techniques of 7 pairs of students answering question l). Coding: Correct answer, partly correct answer, wrong answer, phenomenological based justification, algebraically based justification

Pair #	Pair 1	Pair 2	Pair 3	Pair 4	Pair 5	Pair 6	Pair 7
can A2=B? y-coordinate	Yes	No	No	No	No	No	No

Techni-que(s)	$T_{-T, C\rightarrow 1, V}$ T_{CV}	$T_{T, V, J}$	$T_{T, V, J}$	$T_{T, V, J}$	$T_{-T, C\rightarrow 1, V, J}$	$T_{T, V, J}$	$T_{T, V, J}$
Justification	not justified	Because they are equal in (1,1) where A1 = B, and we cannot find a solution where A2 does not equal A1	Because B always has a distance to the intersection of B and A2's trajectories when A2 is at the intersection	Because they are equal in (1,1) where A1 = B, and we cannot find a solution where A2 does not equal A1	Because they are equal in (1,1) where A1 = B, and we cannot find a solution where A2 does not equal A1	Because they are equal in (1,1) where A1 = B, and we cannot find a solution where A2 does not equal A1	Because B always has a distance to the intersection of B and A2's trajectories when A2 is at the intersection

Analysis

As in the previous two analyses, I will identify differences between the pairs of students, in this case concerning students' use of techniques, and point to relationships to the technical dimension.

In order to solve question K, the required technique is well within the students' capabilities. Table 5 shows that all students who answered the question provided a correct solution by changing the x-value to 2. The only difference between the pairs lies in the use of the slider. All but one pair used it for verification, while four pairs used the slider to justify their solution. Obviously, the students who use the slider for justification have a more developed RC than those who do not. However, this concerns mediation, which is related to the radius of action. Rather, differences in the technical level concern the complexity of the techniques the pairs use in the pragmatic and epistemic mediation toward justification. The students' answers to question K indicate that determining the correct value of the constant term is within their technical level. Pair 1 stand out, however, as they only imagine the technique and do not use the slider for verification.

How can we consider imagined technique in terms of IJ? Imagining data that the technique would produce can be considered a pragmatic mediation. If the imagined data are based on correct warrants, it indicates that the technique is so familiar to the students that they do not need to carry it out. If so, the imagined technique must be well within the students' technical level of problem-solving. If the students also consider imagined data to evidence of a claim, in this case that the value 2 in the x-coordinate results in A2 = B when $s = 2$, then they also imagine verification. In such perspective, the imagined technique of Pair 1 display some level of technical ability toward the goal of justification. On the other hand, the pair is not taking advantage of the feedback that GeoGebra can provide to confirm the results, and do not further justify their answer.

Contrary to question K, students struggle to find a solution to question L. Multiplying with a coefficient without deleting the term generally appears to be at the limits of the students' technical level of problem solving.

Most students (n=5) are inclined to edit only the term and do not consider other techniques. Editing the coefficient is outside of their technical dimension for problem solving. The students who do edit the coefficient (n=2) delete the term first.

The distinction between the pairs' technical level is less clear in this case, as students struggle to find a successful technique. Still, multiplying by a coefficient is more advanced than adding or subtracting. In addition, there is no (visible) coefficient to edit, so the students must have the knowledge of coefficients to use the technique, whereas the term is already created and can be edited. However, the students who use the slider for justification, and hence justify the failure of the technique they used in the algebra view, have a more advanced technical dimension than those who do not, cf. the sequences $T_{T, V, J}$ and $T\text{-}_{T,C\rightarrow 1,V, J}$.

Verification is part of the sequence of techniques for all pairs, apart from Pair 1 for question K who only imagine the technique and the solution. Common to Pairs 2 and 4 in question K is that their sequence of techniques ends with dragging the slider for verification. The students are taking advantage of the feedback GeoGebra provides in the graphic view to confirm the solution. Verification is related to the epistemic value of a claim and is an exercise of RC. In the perspective of IJ, the students do produce data in support of the claim that that A2 = B when the x-coordinate is 2. As we know from Hanna (2000), reasoning has different functions; verification is one of them, explanation is another. It does, however, reflect the difference between pragmatic and epistemic mediation concerning the use of artifacts. Verification only confirms or rejects with respect to an expected solution. The student can only consider why through epistemic mediation. The educational value (Artigue, 2002) of verification is therefore low. As indicated in the analysis of mediation, the explanatory justifications entail intrinsic properties (Lithner, 2008) and have a higher education value.

Pairs 2–7 can also be divided with regard to their justification. One group justify their answer by referring to structural implications of editing the term: "Because B always has a distance to the intersection of B and A2's trajectories when A2 is at the intersection", which is a phenomenological justification.

The other group justify based on the failure of producing a solution where A2 = B: "Because they are equal in (1,1) where A1 = B, and we cannot find a solution where A2 does not equal A1". This is an algebraic justification. In this case, the phenomenological justification is more advanced than the algebraic. Even though both justifications refer to equality as related to the intersection of trajectories in (1,1), the phonological justification relates the applied technique to the distance to the intersection. Hence the evidence for their claim is the structural implication of the technique they used. Contrarily, the algebraic justification uses the failure to produce equality as evidence of the claim. In this case, therefore, the phonological justification is more sophisticated than the algebraic justification.

Based on the analysis, there is no correlation between the use of a particular technique by students in the algebra view and the sophistication of the justification they can produce. It is possible that students can use a technique successfully for problem-solving but not for justification, and vice versa. The slider is a crucial tool in instrumented justification, and students' use of it alternate between verification and justification, which reflect pragmatic and epistemic use respectively. The technical level of students' exercise of RC in IJ is, therefore, related to the sophistication of the justification that the student can consider a used technique, rather than the complexity of the technique itself. This highlights the central importance of using the slider to produce data *and* interpretation through warrants in the students' exercise of RC in IJ processes that include the algebra view.

7.3 INSTRUMENTAL GENESIS RELATED TO RC IN PREDICTION TASKS

In Paper 5, we meet the pair Rio and Lev and explore their IJ process for the first prediction task, capturing their progression through three different techniques: plotting and dragging singular points, tracing points as trajectories of variable points, and drawing trajectories with the pen tool. Their progression of techniques and Rio's conceptual development that unfold in their IJ process is most interesting as a minute window to an instrumental genesis process (Trouche, 2005) in its very beginning. As argued in paper 5, early instrumental genesis is characterized by unstable schemes with irregular behavior, incorrect theorems-in-action, and inefficient rules-in-action that, over time, will stabilize into an invariant behavior. This is rarely addressed in studies that search for and describe the invariant behavior in students' developing schemes (e.g., Roorda et al., 2016) or patterns in schemes across groups (e.g., Pittalis & Drijvers, 2023).

Lev and Rio also participate as a pair in iteration 3, providing the opportunity to follow their instrumental genesis across several prediction tasks.

In this section, I portray how Rio's instrumental genesis progresses in the second prediction task of iteration 2, and the prediction tasks of iteration 3. His instrumental genesis provides a context to discuss how instrumental genesis in IJ processes is also reflected in students' RC.

Rio's justified prediction, test and explanation of D = (s,s)

Rio's and Lev's solution to Q1–3 of the third set (see Figure 18) is analyzed using the IJ model introduced in paper 3 and 5. As in paper 5, Rio is the predominant solver and the most articulate, whereas Lev is a silent observer. Rio is controlling the computer. It is his IJ process and instrumental genesis that we can consider.

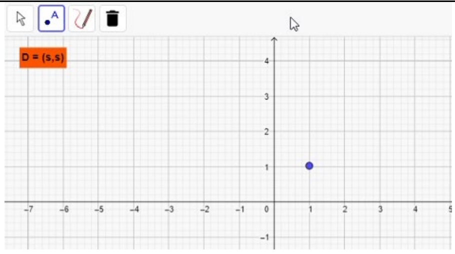
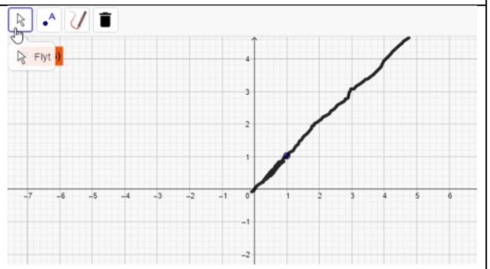
In the analysis, I refer to warrants from paper 5, where the label **WV** covers warrants concerning variables, and **WP** are warrants concerning ordered pairs or points in the coordinate plane. In transcripts, techniques and gestures are given in square brackets and clarifying comments are given in round brackets.

In the first IJ subprocess (see Table 7), Rio forms the prediction in subprocess 1a as Claim 1 (Figure 24), justifies it in subprocess 1b (Figure 23), and forms a written argument.

In the second IJ subprocess, Rio and Lev test the prediction and adjust their prediction leading to claim 2.

Table 8 and Figure 25).

Table 7 – Instrumented justification sub-process 1a and 1b

1	Rio	[Places point in (1,1) with point tool]	
2	Rio	It changes like this [picks pen tool]	
3	Rio	[Draws trajectory from (1,1) → (0,0) → (5,5)]	
4	Rio	Agree?	
5	Rio	Because when *s* is four, then it is also four here [moves cursor to 4 on y-axis, then to (4,4), and then to 4 on x-axis.]	
6	Rio	Drags canvas down and continues to draw a trajectory from (5,5) → edge of view at (6,6)] So, it goes on like this, do we agree?	
7	Rio	It's (*s*,*s*). So, if *s* is zero, it is (0,0), (1,1), (2,2) [while pointing cursor at positions].	
8	Rio	Mumbles (*inaudible*) [Selects the point in (1,1) and then deletes it]	
9	Rio	Reads answer guide and writes: We know that D can only be called (a number, the same number). This means that D, cannot be called (2,1), but can be called (3,3). That's why D moves like this.	

Qualifier: Possible	**Claim 1:** D move as drawn: (0,0) → (5,5) (*infinitely*)

Technique	**Data**	**Warrant**
Pick a value for *s* and plot points with the point tool	If *s* = 1, then D = (1,1) represented by points in the coordinate plane	**WP1:** Ordered pairs of numbers correspond to the x-coordinate and y-coordinate of a point in the coordinate plane **WV1:** A variable should take a random value **WV2:** A variable represents the same value wherever it appears within the same problem
Draw the trajectory of point with pen	Drawn line for D's trajectory: [(1,1) →(0,0), (0,0) → (5,5)]	**WV7*:** The variable can increase infinitely and decrease to 0 **WP6*:** An ordered pair with a variable corresponds to a set of points in the coordinate plane

* The original **WP6** and **WV7** also consider a limit

Figure 24 – Instrumented justification sub-process 1a through the lens of the analytical tool

Qualifier: possible **Claim 1 :** D move as drawn: (0,0) → (6,6) (*infinitely*)

Qualifier: more possible	**Re- Claim:** it is (0,0), (1,1), (2,2)

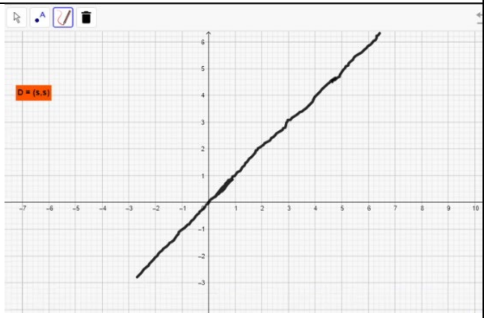
Technique	**Data**	**Warrant**
Draw the trajectory of point with pen	Because when *s* is four, then it is also four here (Line 5)	**WV2**
Moves canvas and draw the trajectory of point with pen	[(5,5) → edge of view]	**WV7*** **WP6***
	It's (*s*,*s*). So if *s* is zero, it is (0,0), (1,1), (2,2) [while pointing curser at positions] (line 7)	**WV2**
Delete plotted point	~~If *s* = 1, then D = (1,1) represented by points in the coordinate plane~~	**WP1** is irrelevant

* The original **WP6** and **WV7** also consider a limit

Figure 23 – Instrumented justification sub-process 1b through the lens of the analytical tool

In the second IJ subprocess, Rio and Lev test the prediction and adjust their prediction leading to claim 2.

Table 8 – Instrumented justification sub-process 2

10	Rio	Construct D = (*S*,*S*) and drags slider for *S* back and forth.	
11	Lev	Exciting	
12	Both	Observe the movement of D moving (-5,-5) ↔ (5,5)	
13	Rio	Actually, we did not quite say that.	
14		[Scrolls back to the GeoGebra app used for the prediction and draws (0,0) → (-3,-3).]	
15	Rio	Otherwise, it does as we thought	

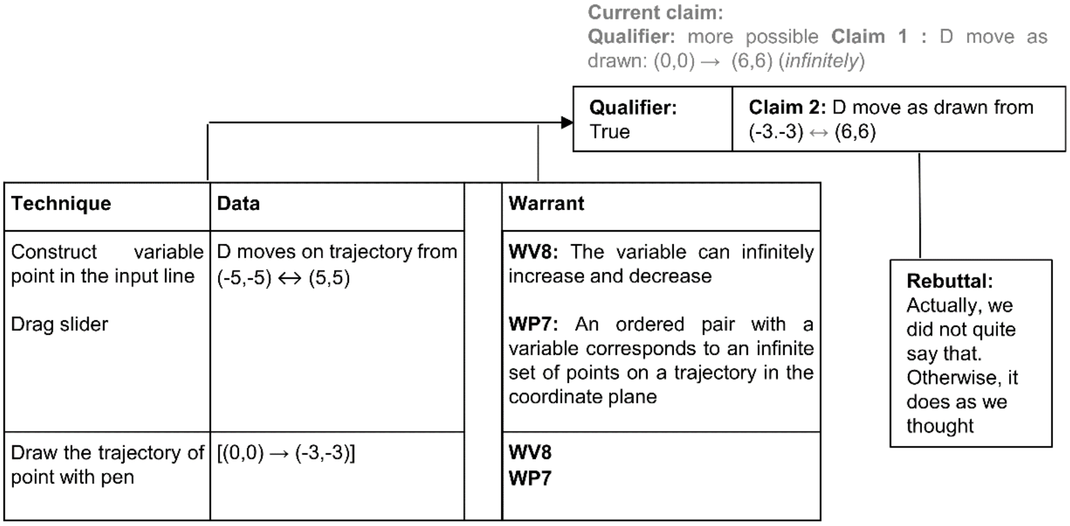


Figure 25 - Instrumented justification sub-process 2 through the lens of the analytical tool

Rio’s instrumental genesis progresses toward drawing the prediction with a pen tool. He starts by plotting a point in (1,1), which is the very same position he regarded as the “staring point” in the first prediction, presented in paper 5. He then continues to draw from this point toward zero, and then toward (5,5) at the edge of view, implying that it continues infinitely in a positive direction. This is coherent with warrants from the previous prediction. Rio then deletes the plotted point. This can indicate that he no longer considers points and warrant **WP1** relevant to the prediction. Rio’s drawing scheme for predicting variable points is stabilizing.

Another indication of Rio’s scheme at these points is that he does not prolong the trajectory of the prediction into the negative space prediction in subprocess 1a and 1b, similar to the prediction in paper 5. Therefore, although Rio did eventually extend the trajectory into the negative region in the paper 5 prediction, Rio has not fully accommodated **WP7** and **WV8** into his scheme yet. In this second prediction task, he is confronted with the trajectory of A2 also moving into negative space in the test, and he adjusts the prediction accordingly. Concerning RC, negative values are not fully part of Rio’s technical dimension, and Rio uses numeric examples to justify the generalized pattern of movement. In the written answer, however, his prediction concerns properties of the variable *s*, as it reoccurs in the ordered pair. In addition, he shows both pragmatic and epistemic mediation of the graphic view and the algebra view toward justification.

Rio’s justified prediction, test, and explanation of A1 =(1,s) and A2 = (1,s-1)

One and a half year later, in iteration 3, Rio and Lev are once again pairing up. As time has passed, they have possibly progressed in both competencies and conceptual knowledge. To be concise, I merely describe their IJ process in predicting, justifying, and testing A1 = (1, *s*) and A2 = (1, *s*-1). See the task in Figure 20 and Appendix B.

In predicting the dynamic behavior of both A1, the pair uses the pen tool for drawing the trajectory in the coordinate plane, including negative values. This time, the drawing shows the marks left as they mimic how the point will continually move back and forth along the y-axis.

In the prediction of A1, Rio claims that “it can only move up and down along the y-axis” and justifies this claim thus: “because it is only the y-value that varies”.

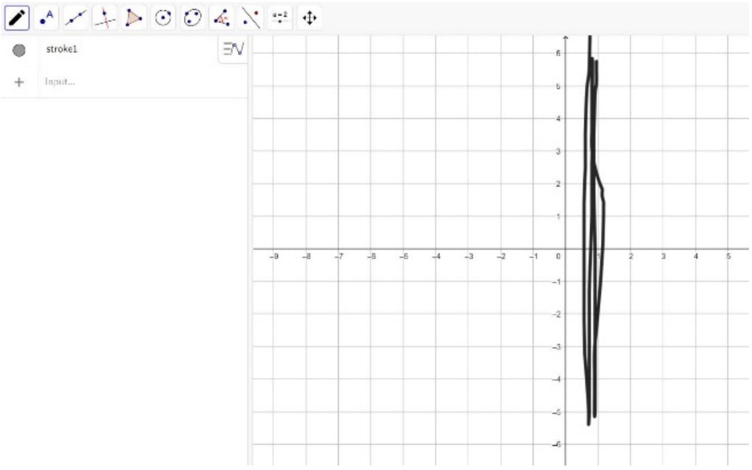


Figure 26 - Lev and Rio’s prediction of A1 =(1,s)

In iteration 3, the students are also asked to draw the trace of the points. This confuses Rio, who argue that point A1 moves infinitely. After testing point A1, Rio realizes that as the variable is controlled by a slider, it is limited to [-5, 5]. Rio then argues that since the y-coordinate is one lower for A2 than A1, so is the trace. They do not draw the traces but construct and test A2 to verify the claim.

The IJ processes in the third iteration indicate a stabilization of Rio’s drawing scheme for predicting variable points. The are no points used in the prediction, and the drawing also expresses the oscillating behavior of dynamic points. Initially, the scheme includes the warrants **WV8:** “The variable can infinitely increase and decrease”, and **WP7**: “An ordered pair with a variable corresponds to an infinite set of points on a trajectory in the coordinate plane”. Those warrants are challenged by the focus on traces in the third iteration. Hence, Rio must adjust his scheme for predicting *variable points* to include the fact that the variable has the limits [-5, 5], and that such limits are particular to the artifact and differ from mathematical theory. In doing so, Rio’s aids and tools competency becomes relevant to him, as he can distinguish between properties specifically related to the tool in use and general mathematical theory.

As for RC, Rio has progressed. His justification no longer includes numeric examples, the properties of the variable as a generalized number is considered sufficient. In addition, the negative space is included effortlessly. This indicates a development in the technical dimension as the sophistication of the argument has advanced. Rio is still able to use both pragmatic and epistemic mediation in relating the graphic view to the algebra view in the test phase, which provides the opportunity for Rio to progress his scheme.

7.4 DISCUSSION, PART 2: RELATING THE SCHEME-TECHNIQUE DUALITY TO STUDENTS RC

This chapter elaborates on the relationships between the three dimensions of RC and students' scheme-technique duality in instrumented justification. In response to the research question, the relationships are summarized and discussed in relation to existing research on justification, the use of digital technologies, and student progression in RC in IJ.

The degree of coverage relates to students' active participation in IJ processes. Active participation implies a shift in students' aim, from problem-solving toward changing the epistemic value of claims facilitated by a digital artifact. Such change is obtained by finding and generating data and warrants. By defining IJ processes, the study contributes with an aspect of RC concerning a concrete form of reasoning. Furthermore, it suggests distinguishing between the mathematical competencies of students working with digital tools by determining the goal of each student's activity.

The radius of action has several relations to students IJ processes. Firstly, it refers to the range of tasks for which students engage in IJ. Prediction tasks and the "equal points" task are examples of the breadth of tasks. Secondly, it involves the contextualization of claims that are being justified. Students engage in IJ by either regarding claims formulated during problem-solving or claims that relate to proposed solutions to mathematical problems. Thirdly, it encompasses the variety of artifacts within the software that students can use for IJ within and across various mathematical domains. Using artifacts for IJ involves the data production through pragmatic mediation and data interpretation through epistemic mediation.

The technical level means the complexity of techniques used or imagined and the sophistication of justifications. It is important to emphasize that the complexity of techniques does not necessarily correlate with the sophistication of justifications and vice versa. The techniques are related to students' schemes and their experience of the efficiency of both rules-of-action and the artifact itself (paper 5). Imagined techniques can reflect a mastery of techniques if students can accurately envision the data they yield. The sophistication of justifications is primarily tied to a student's grasp of underlying concepts and intrinsic properties, as expressed through warrants. Secondarily, the nature of the justification as either phenomenological or knowledge-based (in this case algebraic knowledge) also influences the quality of justification. It is worth noting that in an educational setting, the level of sophistication of verification is comparatively lower than explanatory justification. Therefore, students who use the slider tool for both verification and justification exercise a higher technical level compared to those who only use it for verification. This underscores the importance of a student's radius of action when employing the slider tool in conjunction with other artifacts from the algebraic view.

In IJ processes, students' conceptual knowledge is expressed in warrants when they identify, interpret, and relate the data to the claim through epistemic mediations, which are related to the students' radius of action. Furthermore, the sophistication of justification also relates to students' conceptual knowledge, which concerns the intrinsic properties to qualify the mediation and how they are used in inference between *theorems-in-action*. Here we see how two dimensions, the radius of action and the technical level, are related through students' epistemic mediation expressed in warrants. This relationship is in accordance with IAME, where students' knowledge of concepts, as well as artifacts, is considered influential to their use of a tool (Trouche, 2005), and it concerns the scheme-technique duality.

Recognizing the connection between students' scheme-technique duality and the three dimensions of RC leads to an important inquiry: How can we aid students in developing their RC in IJ processes? To address this, I will explore different perspectives on students' advancement in IJ, some of which stem from issues raised in the papers.

In paper 5, I address the critical issue concerning students' conception of variable points:

> ...the translation of representation, such as the one-to-one mapping of terms, which Duval (2006) problematised in relation to dynamic environments. In this case, how do the students perceive A and B in the final prediction? Do students perceive A and B as particular points that move, or have they objectified A and B as structural patterned movement or, possibly, a hybrid of the particular and generalized? Such questions could be addressed by observing the students' progression in the prediction of other variable points. (p. 24)

Students' development toward a generalized conception of variable points in the exercise of RC concerns their technical dimension. By analyzing Rio's instrumental genesis throughout the prediction tasks, there are three stages in the conceptual progression from a discrete to a continuous understanding of variable points.

At the initial stage, students begin by assigning a value to the variable and creating individual points that can be manipulated across the coordinate plane to assume other values of the variable. This scheme, with its discrete notion of variable points, uses numerical values as data to support a claim about structured, patterned movement. This stage is crucial as it allows students to gain experience with "the one-to-one mapping of terms", which Duval (2006) highlighted as potentially challenging for students in dynamic environments. Therefore, the first stage plays a pivotal role in establishing a coherent understanding of the variable as a generalized number, laying the groundwork for reasoning about variables.

As students' progress, they reach an intermediate stage where the scheme incorporates a hybrid of continuous and discrete conceptions. This is evident in Rio's prediction that point D = (s, s). While numeric positions of points are still used as data, the structured movement is considered continuous in the drawn trajectories. Rio constructs a point but has no further interaction with it, and the actual prediction is drawn with the pen tool. However, the justification still involves data on numeric values and positions. This stage is supported by the graphic representations of variable points, where the variable point is represented both by an actual point on the coordinate plane and its structured, patterned movements. Therefore, variable points serve as a stepping stone for students to justify their reasoning with regard to variable properties, a unique learning opportunity that is not readily available in traditional pen-and-paper environments (Noss et al., 2012).

The third stage marks a significant milestone in students' development toward a generalized conception of variable points. At this stage, students represent variable points by drawing infinite or limited trajectories. This scheme, with its continuous understanding of variable points, uses the variable as a generalized number to support a claim about structured, patterned movement. We observe this third stage in Rio's prediction that A1 = (1, s). Advancing to the third stage requires the ability to discern the constraints of the slider tool on the variable points in GeoGebra from mathematical theory, indicating a deeper understanding of the concept.

In instrumental genesis, the student's knowledge of an object and its artifact influences the progression of instrumental genesis in IJ processes, which also leads to a progression in the student's conceptual understanding and RC. In paper 5, I demonstrate how instrumental genesis in IJ processes is advanced due to the inefficiency of applied rules-of-action (Vergnaud, 1998b) and the constraints of an artifact, requiring it to be consistent with effective rules-of-actions to produce data. It is important to note that what is considered efficient or inefficient is relative to the student. To facilitate the progress of students and support their conceptual understanding, it is necessary to address this relativity and encourage their progression from techniques they still consider effective.

In the equal points task, many students limited themselves to using only one technique in question L, resulting in a lack of progression in their instrumental genesis. As explained in paper 4, when students realize that a technique did not produce equality, they gave up and did not attempt to find a different solution. This led to the justifications of Pairs 2, 4, 5 and 6 in question L (Table 6). The question is at the limits of the students' technical abilities, so how do we support them in progressing, rather than resorting to one single technique?

From a task design perspective, how can we support students who are at their limits of their technical abilities in progressing, instead of relying on one technique? On two occasions during classroom experiments, I caught students failing with their technique and urged them to try another technique by hinting that it is possible for A2 = B, indicating the value of the claim.

Implementing such a solution in a classroom setting would be impractical as it would require the teacher to closely monitor students in order to intervene at the appropriate time. Nevertheless, the approach demonstrates that indicating the epistemic value of a claim can prompt students to reflect on the inefficacy of their technique or selection of artifact. From the perspective of task design, it may be advantageous to include an indicator of epistemic value. Instead of asking "Can A2 = B if you change the y-coordinate?", a question indicating an epistemic value could be: "By altering the y-coordinate of A2, A2 can equal B in a single point. How many solutions can you find? Explain why your solution(s) lead to equality." This way of posing a question adheres to the direct wording of the questions implemented in iteration 3, which directs students' attention toward fruitful actions without providing the answers. Additionally, it emphasizes the significance of forming questions, particularly when learners are grappling with challenging concepts.

Both KOM and IAME have an individualistic perspective, which can sometimes conflict with the more communal approach to reasoning and justification (e.g., Jeannotte & Kieran, 2017; Yackel & Cobb, 1996). Typically, arguments are evaluated based on generally accepted truths, or key ideas within the community, whether it be the classroom or the broader mathematical community. However, IJ takes a different approach. Instead, it considers the change in epistemic value from the perspective of the student, aligning with Lithner's (2008) description of reasoning as

> the line of thought adopted to produce assertions and reach conclusions in task solving. It is not necessarily based on formal logic, thus not restricted to proof, and may even be incorrect as long as there are some kinds of sensible (to the reasoner) reasons backing it. (p. 257).

The IJ process places a crucial emphasis on the knowledge and interpretations of the individual student. This necessitates a student-centered approach that values the student's rationales over their adherence to mathematical theory. IJ provides an analytical tool that map how students pursue the goal of justification while using tools. This student-centered perspective advocates for an inclusive view of students' use of tools as an exercise in RC.

Hence, the approach taken by IJ acknowledges the epistemological gap, identified by Sabena et al. (2014), where teachers may expect a theoretical argument, while students approach their answers experimentally. The student-centered approach embraces students' inclination to rely on empirical or phenomenological knowledge. However, this perspective has its limitations, as it only allows for justifications based on data and evidence that the students can produce themselves. This underscores the importance of the communal aspect of justification as a practice. Other positions on justification (e.g., Dreyfus, 1999; Jeannotte & Kieran, 2017; G. J. Stylianides, 2008; Wood, 1999; Yackel & Cobb, 1996) emphasize that certain types of knowledge and arguments are more valuable than others, and that developing the communal understanding supports the development of students' RC. Therefore, when supporting students' progress in justification, it is crucial to consider the communal perspective.

The notion that IJ is solely individualistic may be nuanced when considering data analyzed with the IJ tool, which involves pairs of students. This naturally resulted in students sometimes disagreeing or having to explicate their reasoning and understanding. In applying the analytical tool, this was handled by labelling warrants as specific to a certain student (examples are found in papers 3 and 5). In those cases, the resulting data are the same for each student, but the warrants by which the data are perceived and interpreted as evidence differ between the pairs of students and are negotiated between students.

Thus, concerning the sophistication of justification, the warrants must be challenged and negotiated in the communal context. Some have suggested that having students observe invariance while they manipulate objects in DGAEs (e.g., Olive et al., 2010) can potentially progress them from experimental to theoretical mathematics. However, if the warrants must be challenged, such observations alone are not sufficient to advance to theoretical justification. Future research on IJ could explore how students' warrants can be challenged so that the sophistication of their justification progresses.

In conclusion, this chapter has shed light on the complex relationships between students' scheme-technique duality and the three dimensions of RC in instrumented justification. Moreover, it highlights the importance of conceptual knowledge and epistemic mediations in students' development of RC in IJ processes. By identifying, interpreting, and relating the data to the claim through epistemic mediations, students can express their conceptual knowledge in warrants and exercise their RC.

Finally, this chapter asks how to aid students in developing their RC in IJ processes. By exploring different perspectives on students' progression in IJ and addressing critical issues related to their conception of variable points, we can better understand the challenges and opportunities for improving students' performance in this area.

8 LINKING RC AND IAME

This chapter addresses RQ3:

> Which theoretical links can be established between Reasoning Competency and the Instrumental Approach to Mathematics Education from the theoretical developments of the study?

This chapter consolidates the various theoretical considerations explored throughout the study and elaborates the theoretical developments, struggles and transitions in focus. The objective is to expound the progression of the analytical perspective (McKenney & Reeves, 2018) of the design research process. This will consider the broader context of networking for advancing KOM and IAME by suggesting potential links between the two perspectives.

The chapter falls within four parts. 8.1 elaborates the rationales for the developments of the analytical tool for IJ, which assist in linking the two frameworks in a networking perspective. 8.2 describe the development an analytical tool, and 8.3 elaborate on further refinements of this tool, including discussions of how the developments are potential links for networking. 8.4 are the final part of the discussion, addressing the research question and revisiting notions from the theoretical framework.

The theoretical developments are illustrated progressively, and the reader should be aware that each illustration only conveys a particular focus at a particular moment in progression to visualize the points made.

8.1 A NEED FOR AN ANALYTICAL TOOL

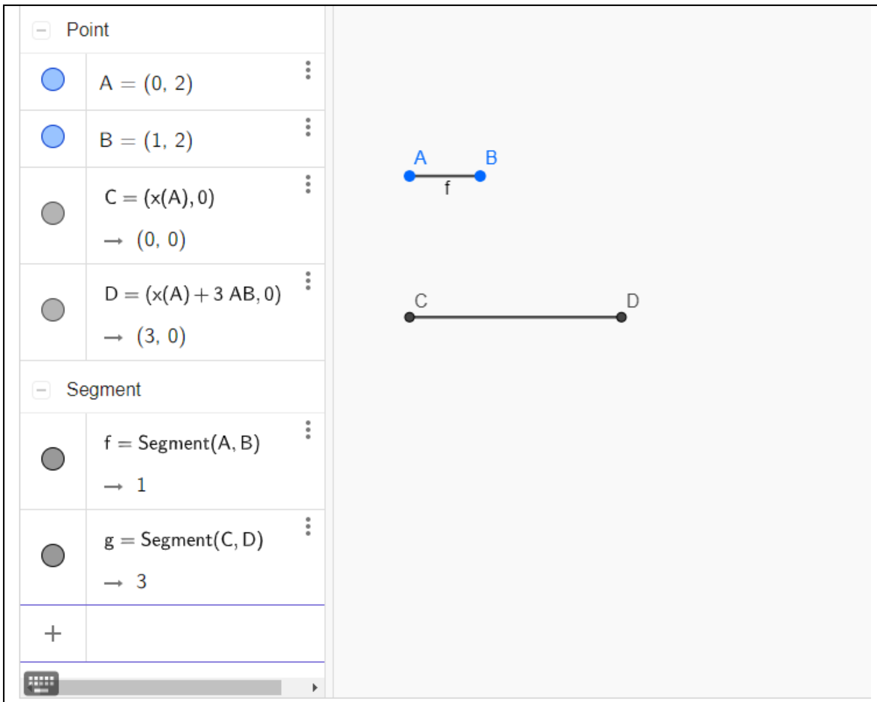
In the first iteration of education design research, the need for an analytical tool emerged through a parallel analysis of the students' solutions to tasks, operationalizing the IAME and the KOM framework. A parallel analysis means to apply (at least) two different theoretical perspectives to the same set of data and discuss how each perspective illuminates the research problem and relates to the other one (Maracci, 2008). It is a method within the *coordinating and combining* categories of NT strategies (Prediger, Bikner-Ahsbahs, et al., 2008).

The aim of the parallel analyses was to capture students' work with tools in justification, and the hypothesis was that the students tool use would differ according to competencies. However, such difference did not emerge in these analyses, emphasizing the need for developing the analytical tool that accentuated justification processes. The following subsection 8.1.1 argues this point by presenting a parallel analysis as a departure for discussing the need for an analytical tool.

8.1.1 An example of a parallel analysis

First, I describe the case subjected to a parallel analysis, and then, how KOM and IAME were operationalized in this analysis.

The case involves a pair of students working on one of the explorative tasks presented in Chapter 6, Figure 13: “Relationships between lines”. Figure 27 includes question 4 and 5 of the task and shows the state of the GeoGebra app with which the students were answering the questions. The utterances were transcribed and their interaction with the tool described.



Q4) Is it true that CD always will be three times the length of AB? Explain why this might (not) be true. What is your argument?

Q5)

Construct a line EF that is always half of CD. Argue why your solution is correct.

Figure 27 – The state of GeoGebra as students S1 and S2 answer Q4

In the analysis from the KOM perspective, the competency that students exercised were identified. Students’ exercise of RC was specified as students putting forward claims and justifying them by drawing inferences, thereby altering the epistemic value of their claims (Duval, 2007). Hence, claims and changes in epistemic values were identified. Further, students’ justifications were characterized as, e.g., exemplification, generalization, or verification by feedback. This corresponds to an evaluation of the student’s degree of coverage (Niss & Højgaard, 2019).

In the analysis from the IAME perspective, the students’ tool use was described with regard to the artifact and the activity, such as dragging, typing etc. The tool use was also classified as a case of either epistemic or pragmatic mediation. It was expected that changes to the epistemic value of a claim would be related to epistemic mediation (Misfeldt & Jankvist, 2018).

In the following, I present an example of a parallel analysis of a transcript of students’ utterances with a description of their interaction with GeoGebra.

Then follows an analysis of each coding, and finally, a discussion of how they relate to one another. A condensed version of such a parallel analysis is presented in Gregersen (2020).

Transcript	KOM RC: Reasoning Competency RepC: Representation competency PHC: problem handling	Claim, epistemic value	Mediation	Artifact, object, activity
ST1: [Drags point B so that AB is very short, then a little longer] "What if I do like this?"			Pragmatic	Graphic view, point, dragging
ST2: Have a look, it is three times as long... it is like three pieces... (pointing at segments)	RC: visual comparison of segments (example)	CD = 3AB, possible	Epistemic	Graphic view, line segments, gestural activity
ST1: Yeah				
ST2: Yes, I would like to show that it is three, but...	RC: searching for other inference possibilities than visual comparison	CD = 3AB, possible		
ST1: The short one is 33 % of the long line	RepC: translates from fraction to percentage			
ST1: Okay, [typing answer] yes, it is three all the time.	RC: Generalizing based on visual comparison	CD = 3AB, likely		
ST2: It should be so easy, and we are just not able to find it. [returns to GGB and drags point B]	RC: Searching for inference possibilities		Epistemic	Graphic view, point, dragging (possibly also algebra view)
ST1+2: [Both looking at the screen.]	RC: Searching for inference possibilities			
ST1: I think I worked it out! I think that maybe this one here. Because if you look at D [pointing at point D in the algebra view] it might be that it is three times larger than A and B. And it is influenced by, what is it... I just have to check D [Drags point B, D moves]. It is influenced by point B. Yes. And also point A [Drags point A, C and D moves]. And this one [*point C*]. It is only influenced by point A.	RC: Inference drawn between definition of D and length of CD RepC: Identifying relationships between representation	CD = 3AB, more likely	Epistemic Epistemic	Algebra view, gestural activity Graphic view, points, dragging
ST2: Oh, so we see that D is influenced by A and B, and C is only influenced by A. This is true.	RepC: Identifying relationships between representation		Epistemic	Algebra view and graphic view, Points
ST1: The first (referring to the point E), that should be C then? Oh, I think I got it, are you ready?	PHC		Pragmatic	Algebra view, points, input field,
ST2: We need to do point E and F				
ST1: [types () in input field]	PHC		Pragmatic	Algebra view, input field

ST2: You need to write E =	PHC		Pragmatic	Algebra view, input field
The students then attempt to construct EF by copying the structure of C and D. They discover that the inputs written in the algebra view are coordinates but fail to construct EF and continue to the next question.				

In the students' exercise of RC, they agree with the claim that CD = 3AB, based on a visual comparison of AB and CD. ST2 expresses that the visual data are insufficient and searches for other inference possibilities. First, the student has no luck, so the visual comparison of CD and AB is generalized to be "at all times". Then, ST1 notices "3AB" in the definition of D and states that "it might be that it is three times larger than A and B". ST1 do seem to use "D" as a label for the length of segment CD, since D is what varies. The epistemic value of the claim increases each time it is asserted. Then the students shift to investigating the symbolic representation in the algebraic view as related to the graphic representation. When the relations are established, the students move on to question 5 and problem solving. They do not draw conclusions in relation to their claim.

Concerning IAME, the students primarily show epistemic mediation. They use dragging of the end points of lines to compare the relationships among them and to generalize the comparison. Similarly, dragging is also used to establish relationships among the representation of the algebra view and the graphic view. Epistemic mediation of the algebra view is accompanied by gestural activity or verbal expressions pointing towards information in the algebra view. Once the students turn to question 5, they interact with the algebra view through pragmatic mediation.

Relating the two analyses, what can we say about RC and tool use? Whenever students exercise RC, epistemic mediation occurs to obtain information that relates to the claim. The epistemic mediation occurs in conjunction with dragging, observation of points' behavior in the graphic view, or enunciation of information in the algebra view. However, the students also use epistemic mediation when exercising representation competency, but this time in the context of establishing relationships between objects. Despite epistemic mediation not being specific to RC, epistemic mediations are involved in both the formation of claims and changing the epistemic value of a claim. Though it provides some descriptive qualities (Prediger, 2019), identifying the pragmatic and epistemic mediation provided little new insight into how the epistemic mediation influenced the epistemic state of a mathematical claim.

However, the parallel analyses did indicate that the information students obtain through epistemic mediation, exercising RC or representation competency, concerned different processes. When exercising RC, the epistemic mediation concerned students' own inferences toward the change in epistemic value of a claim, related to goal justification, while representation competency concerns establishing relationships between constructed objects.

Therefore, in IAME the exercise of different competencies can possibly be related the goals that orient students' instrumented acts and are part of students' schemes (Vergnaud, 1998b). Figure 28 illustrates the two theoretical perspectives of the parallel analysis with KOM in red, and IAME in blue. The aspect of justification within RC is understood as an inference that changes the epistemic value of a claim. The IAME perspective shows the fundamental construct of instrumental genesis along with epistemic and pragmatic mediation. At this stage of the study, the link suggested between goals are illustrated with orange text and an arrow illustrating the connection. The arrow connects students' schemes that are goal oriented with the change in epistemic value. Such a connection concerns the relationship "students-activity" in relation to tool use.

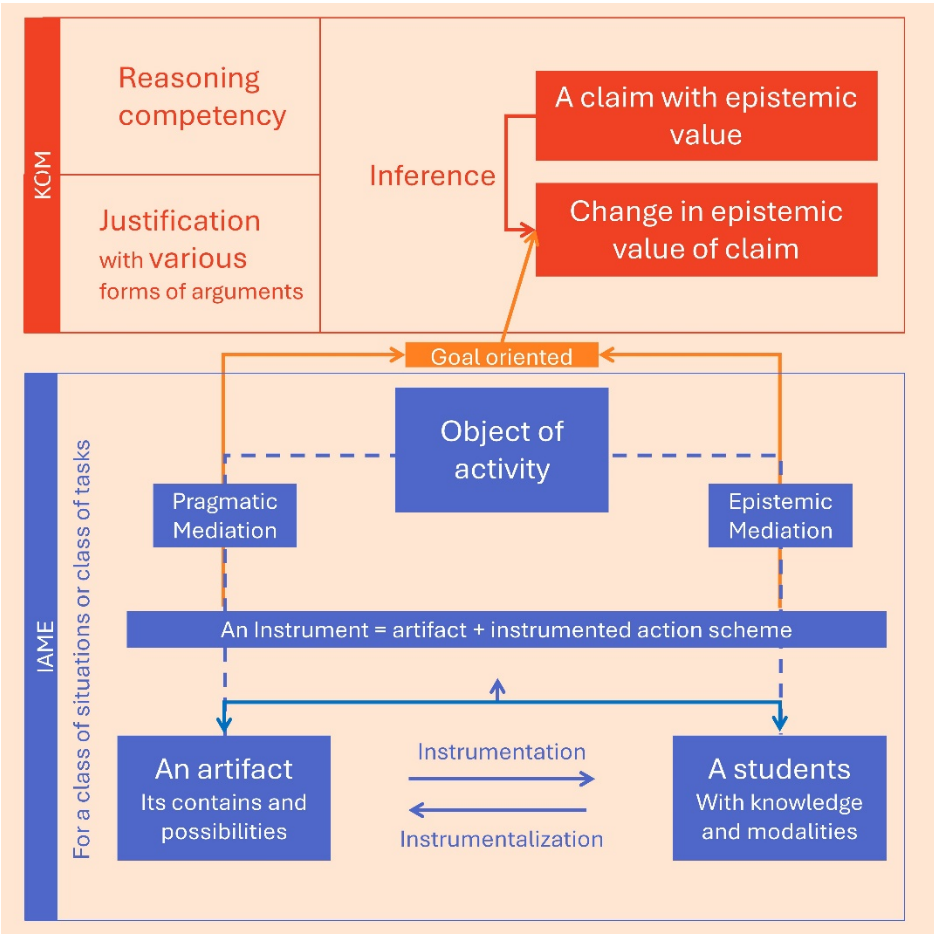


Figure 28 – Illustration of theoretical perspectives in the parallel analyses and suggested links. KOM is red, IAME is blue, and links are orange

To further understand how this goal unfolds in students' tool use, it seemed that other notions or frameworks concerning justification and justification processes were needed, since both IAME and KOM lack local conceptual elements particular to justification and justification processes. Several frameworks were considered, and eventually Toulmin's (2003) model was used. Below, I will briefly describe how.

Though Toulmin's model originates from outside of MER, it has a rich use history in reasoning within MER, but it has also been adapted to other fields, demonstrating its applicability across sciences. In Toulmin's model, a claim is a statement made by the speaker, with a qualifier to indicate its likelihood. Justification of the claim is established through other elements of the argument, such as data, warrant, and backing. Toulmin's model is a construct based on an assumption of the elements that make up arguments. It rests on hypotheses about standards, values, and convictions in jurisdiction and practical everyday reasoning (as opposed to logical and philosophical standards), taking into account the complexities and uncertainties inherent in human reasoning. The model's adaptability is evident as "*certain constant field-elements can be discerned in the way in which argumentation develops*" (Toulmin, 2003, p. 2). Assuming that Toulmin has successfully drawn out such elements, these must also be present when students exercise RC using tools.

Through close collaboration with Anna Baccaglini-Frank, an analytical tool was developed by reinterpreting Toulmin's (2003) model with respect to IAME and KOM. Its development is described in the following section.

8.2 THE DEVELOPMENT OF AN ANALYTICAL TOOL FOR INSTRUMENTED JUSTIFICATION

Papers 2 and 3 demonstrate the evolution of the analytical tool for instrumented justification. Paper 2 introduces an initial version of the tool (see Figure 29), while paper 3 showcases the developed version used in the remainder of the study (Figure 30). The analytical tool and its application is explained in section 7.1 and in the papers. Here, I address theoretical considerations, elaborate on the rationale of the development, and highlight their significance in linking KOM and IAME. In conformance with design research, the tool was developed through a continuous process of theoretical reflection and retrospective analysis of students' work.

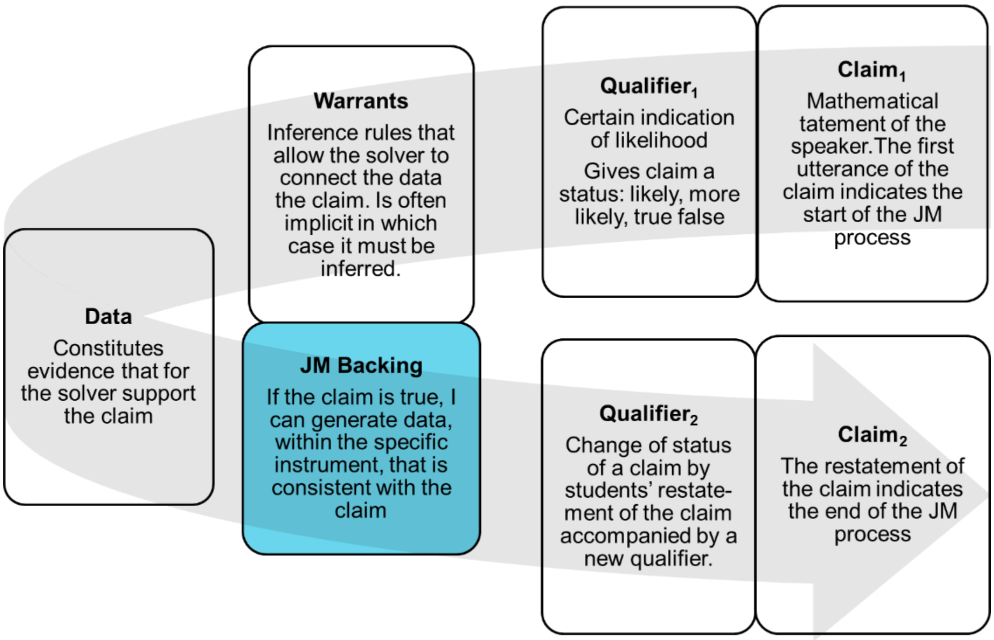


Figure 29 – Generalized version of the initial analytical tool, presented at "Matematikdidaktikkens dag" 2020, Emdrup, KBH (Gregersen & Baccaglini-Frank, 2020)

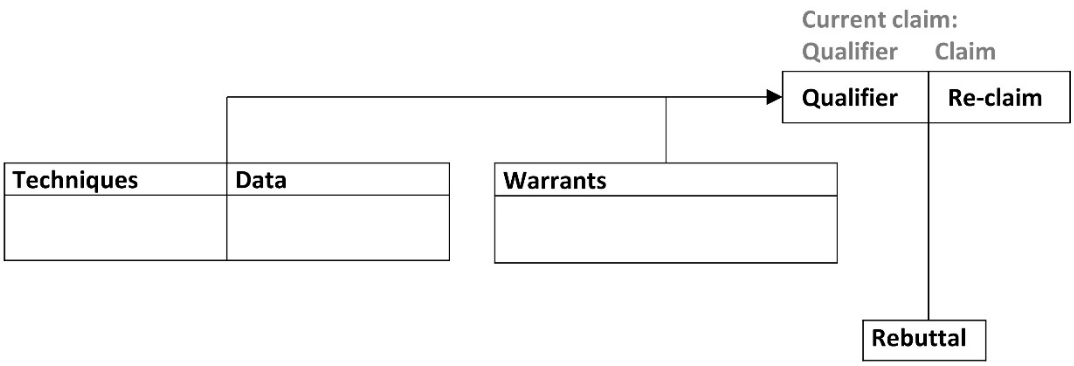


Figure 30 – Illustration of the analytical tool for instrumented justification, paper 3

Figure 31 illustrates the theoretical advancements made from paper 2 to paper 3. KOM is represented in red, IAME in blue, and the analytical tool with elements from Toulmin's model is represented in green. Links between elements are indicated in orange. What the illustration also conveys is that other constructs from IAME are emphasized compared to Figure 28. Note that mediation is no longer emphasized, and neither is the object of activity.

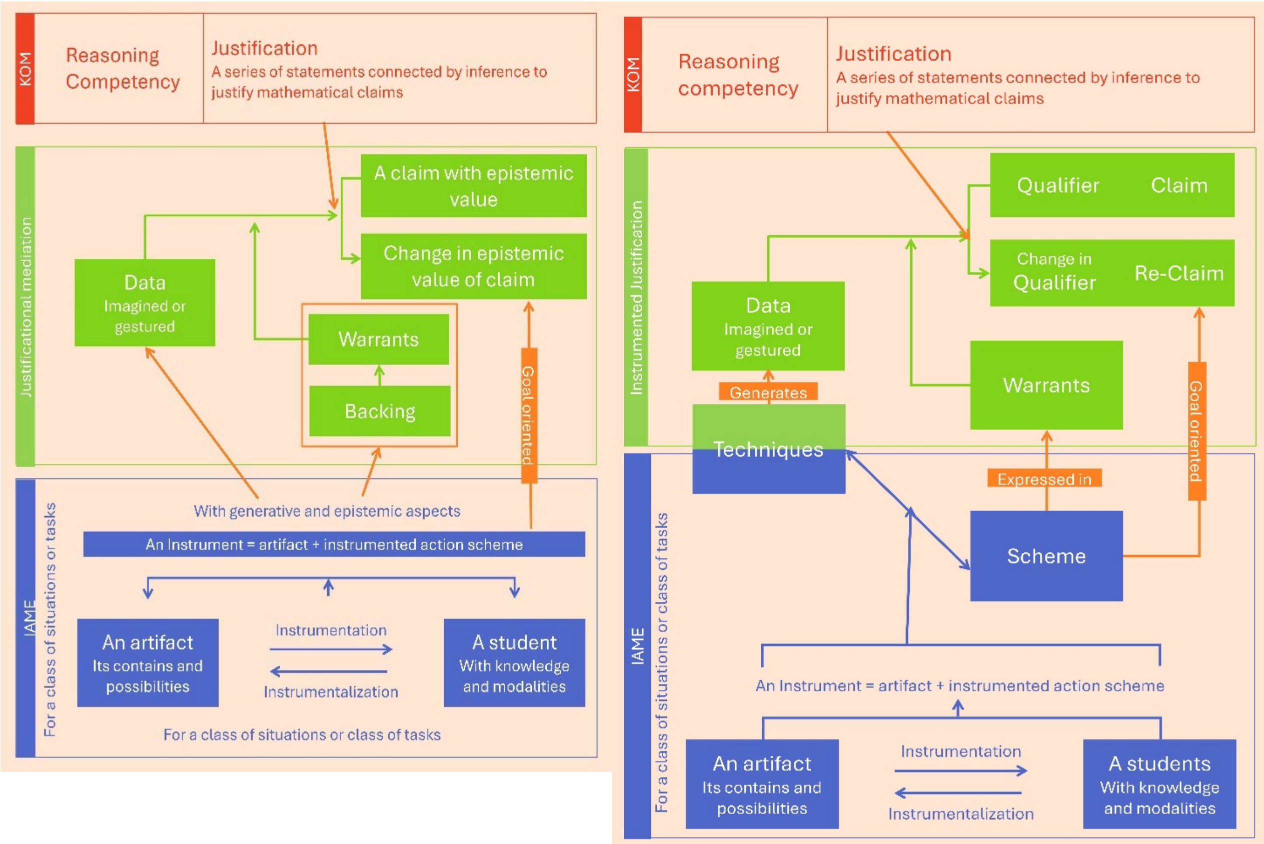


Figure 31 - Illustration of theoretical links established in a) paper 2 and b) paper 3. KOM is red, IAME is blue, the analytical tool is green. Links between perspectives are marked with orange arrows and text

Paper 2 introduced the analytical tool as justificational mediation (JM), but upon further consideration, some issues were identified with this description. Mediation "comes from the subject's activity being oriented toward the object of the activity" (Rabardel & Bourmaud, 2003, pp. 668-669) and can be either pragmatic or epistemic. However, in applying the analytical tool, the change of the epistemic value of a claim involves multiple mediations, both epistemic and pragmatic, oriented toward several objects. The notion of justificational mediation proposes an overarching type of mediation consisting of multiple other mediations. However, this deviates from the original conception of mediation and could potentially cause confusion instead of clarity. Instead, in paper 3, the analytical tool leads to the following definition of *instrumented justification*:

> *Instrumented justification is a process through which a student modifies the qualifier of one (or more related) claim(s) using techniques in a digital environment to generate and search for data and warrants constituting evidence for such claim(s).* (p. 135)

The definition of IJ remains consistent throughout the study. The crucial part is that IJ is a process, and the analytical tool suggests how to identify this process. In paper 2, this is introduced in the reinterpretation of the Toulmin model's analytical unit, which shifts from a finalized argument to the process of changing the epistemic value of a claim. This process relates to the qualifier in Toulmin's model. In paper 2, we state that "The first utterance of the claim indicates the start of the JM process, in which the aim is to change the qualifier" (p. 453), and "We recognize such a change of status of a claim by students' restatement of the claim accompanied by a new qualifier." (p. 435).

The adaptation of the analytical unit creates a link to KOM (note the arrow from KOM to analytical tool in Figure 32), where RC is defined as the production or analysis of arguments that support mathematical claims (Niss & Højgaard, 2019), and considers the process perspective of justification. Additionally, it recognizes that instrumental genesis is a process. The new unit of analysis affects all the elements of Toulmin's (2003) model that originally depicted a finalized argument, and which now contains several units. For instance, the data and warrant elements incorporate all data and corresponding warrants produced in the unit of analysis, which collectively leads to the change of the qualifier. In paper 3, the term *re-statement* is redubbed *re-claim* in line with Toulmin's terminology. A re-claim may take the form of a partial recapitulation or an indirect reference, such as "it". For instance, a statement like "it must be true" is considered a re-claim accompanied by a qualifier. In paper 3, the epistemic value of a claim is now referred to as a change of qualifier, in line with Toulmin's terminology. The significance of the qualifier and how it was derived from the data are elaborated upon in paper 3:

> The qualifier can then be inferred from the student's actions; for example, a statement can be uttered with hesitation, or if a student continues to search for data, we can infer that the student is not yet convinced that the claim is true. The qualifier can change from "possible" to "more possible", "less possible", "true", or "false". (p. 121)

The advancement in the inferences about qualifiers was achieved by using the tool in a wider range of student cases, which presented more complex situations where the qualifier was not explicitly stated. Another adjustment introduced in paper 3 was that most IJ processes consist of multiple restatements, at times with rebuttals that result in new or revised claims. Thus, the analytical tool was reevaluated to encompass these more intricate processes as IJ sub-processes.

While paper 2 emphasized the importance of backing as a core component of JM, paper 3 omitted this aspect. In paper 2, we highlighted that:

> In the context of JM, we consider the backing to be an explanation of why the warrant is relevant (Simpson, 2015). Central is, that the aim of JM is to change the status of the claim, so the backing must explain why the warrant is relevant for generating data that allows the change in the status of the claim. Thus, the backing becomes fundamental to the JM process. Currently, we have reached the following formulation

> of backing in JM processes: If the claim is true, I can generate data, within the specific instrument, that is consistent with the claim. This seems closely related to Vergnaud's (2009) notion of theorem-in-action, a sentence that the solver believes to be true, but that may in fact be false. (p. 453)

However, in paper 3, we no longer analyzed students' backing, for various reasons. The application of the backing remains unclear across MER (Simpson, 2015), leading to low replicability of inferences about student backings. In paper 2, we inferred a backing, which we elaborated as a possible theorem-in-action: "If the claim is true, I can generate data, within the specific instrument, that is consistent with the claim." In this case, the backing acts as a meta-theorem-in-action, referencing the trustworthiness of digital artifacts and the data they process. In other words, it concerns student trust in the tool as an instrument consistent with mathematical theory in the generation of data. However, in hindsight this meta-theorem-in-action was imposed on the participating students, since they were specifically asked to use GeoGebra. Finally, the nature of students' arguments caught our attention in the case we analyzed for paper 3. Altogether, in paper 3, we shifted our focus towards the techniques and generation of data. I will touch upon backing as a link to KOM in the discussion.

An essential development from paper 2 to 3 was the shift from considering schemes in its generality, as generative and epistemic aspects, to the use of the scheme-technique duality (note the arrows from the blue IAME constructs to the analytical tool in Figure 31).

In paper 2, we wrote:

> The generation of data is the product of the generative aspect of the schemes used (e.g., dragging, creating objects on the screen and interacting with them, utterances and other hand-gestures) that are carried out by students. Warrants are the epistemic aspect of the schemes used. (p.453)

While in paper 3, we wrote:

> A second feature of our analytical tool is that a technique frame appears next to the data. This is because the main source of data, as students attempt to justify claims in a digital interactive environment like GeoGebra, is the effect of their use of techniques (as described in the TIG). The invisible schemes direct and organize actions with or on the data, but they also contain conceptual elements and rules that regulate actions (Drijvers et al., 2013). Such rules can be seen in the model as warrants, which are inference rules that connect the data to the claim. (p. 122)

An advantage of using the scheme-technique duality (Drijvers et al., 2013), discussed in subsection 2.6.1, is its ability to establish direct correlations between observed techniques, the resulting data, and the inferred warrants. Considering the analytical tool in Figure 30, the reader might get a sense of this correlation. The introduction of the duality also emphasizes a double orientation of the warrants to both direct the generation of new data and interpret the data as evidence for or against the claim, or even regard the data as irrelevant to the claim.

As such, the scheme-technique duality (Drijvers et al., 2013) provides a pair of concepts to determine observable interaction with a tool as techniques that are organized by a scheme. Those schemes, in turn, are also expressed in the warrants that interpret the outcome of those actions and conduct inferences. By doing so, the analytical tool establishes links to the IAME that allows a clear identification of how students' interactions with the tool contribute to the change of the epistemic values of claims through the generation of data.

8.3 Refining the analytical tool for instrumented justification

The preceding section described the development of the analytical tool and discussed how it can link IAME and KOM. Paper 5 and Chapter 7 in the kappa refines the analytical tool with concern to components of students' schemes, mediation, and the three dimensions of RC, establishing additional links between the RC and TIG. Subsection 8.3.1 focus on the contribution of paper 5 and the components of schemes, while 8.3.2 focus on mediations and the three dimensions of RC.

8.3.1 The components of scheme

As portrayed above, the IJ tool establishes links to both KOM and IAME through the scheme-technique duality. Paper 5 elaborates further on the components of scheme (Vergnaud, 1998b), that is the operational invariants rules-of-action, theorems-in-action about concepts-in-action, and the possibilities of inference. Those constructs are operationalized in further analysis of an IJ process to relate students' conceptual knowledge to their RC which is illustrated in Figure 32.

Central to this is the link between warrants and schemes: "schemes are goal-oriented concerning the task at hand (Vergnaud, 1997)—in this case, the goal is putting forward a *prediction* and *justifying* that prediction by changing the epistemic value of the prediction. Such activity involves both rules of-action and theorems-in-action about relevant concepts-in-action" (paper 5, p. 66).

Figure 32 includes the components of schemes. The orange arrows and text indicate how the components are related to the existing elements of the analytical tool. I will explain these relations one by one.

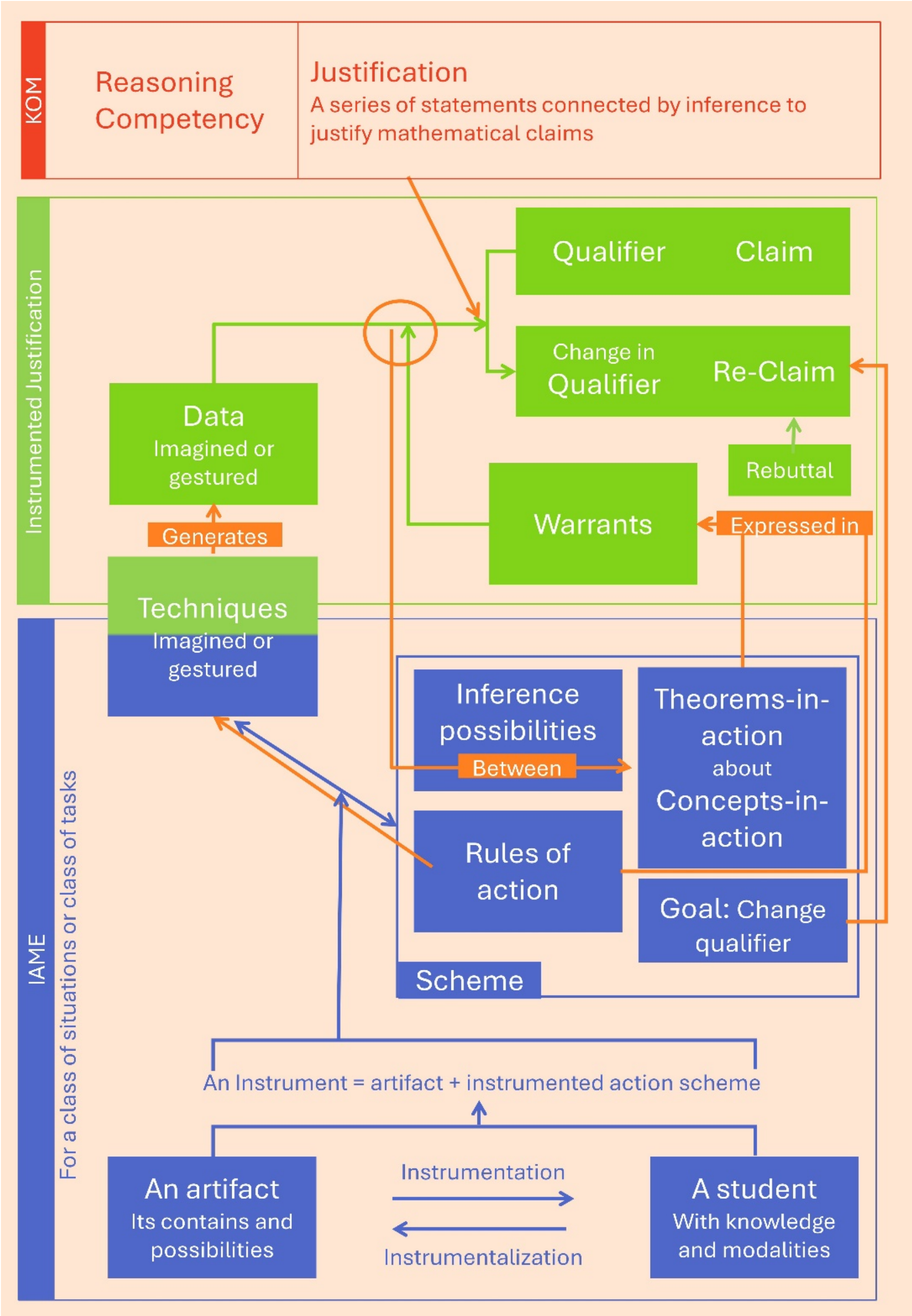


Figure 32 – Links established between the scheme-technique duality and RC through the analytical tool for IJ. The circle marks the point in the process where warrants interpret data as evidence for or against the claim, which allows for inference between operational invariants

Schemes have a component that consists of goals and sub-goals, and which depends on situational variables (Vergnaud, 1998b). It has already been established that in IJ processes, the goal is to change the epistemic value of a claim, and KOM describes that this is done through inference. The subgoals are not elaborated in paper 5, but they could be along the lines of: "Make the trace start further down the x-axis" or "Explore if and how the new data aligns with the claim".

Warrants are inference rules that connect the data to the stated claim. In paper 5, those inference rules are considered as either rules-of-action or theorems-in-action. Rules-of-action, the less complex of the two, rely on operational invariants. They are the generative aspect of a scheme, governing and organizing the techniques that students use to generate data in support of a claim. A defining feature of rules-of-action is their effectiveness. In the context of IJ, effectiveness can be seen as relating to the data that a given technique generates. This notion of relevance is intertwined with the perspective that justifications must be grounded in the intrinsic properties of the task. This, of course, extends to the techniques and rules-of-action employed by students in the IJ process. Ultimately, the RC in IJ processes also includes student generation of relevant data.

In Figure 32, note the circle formed by the intersection of the arrows between data and warrants. This intersection is significant because it marks the place where warrants interpret data. From there, an orange arrow points towards inference possibilities, a component of schemes. Inference is a defining characteristic of RC, making inference possibilities particularly relevant to study. In paper 5, I argue that the inferences made in IJ processes are between the operational invariants made in the interpretation of data, and we can draw assumptions about these inferences by considering students' warrants. This leads to a more complex component: the operational invariants, which are propositions about concepts-in-action held to be true (Vergnaud, 1998b).

Paper 5 suggests that warrants that are not rules-of-action can be considered as theorems-in-action. In IJ processes, those warrants evolve as students produce more elaborate data or realize the relevance of concepts or properties. As explained in paper 5,

> The change in epistemic value occurs through an interplay of producing data and interpretation through inference between the operational invariants. The inference allows the production of additional supportive or contradictory data. This cycle continues until the epistemic value is changed. (p. 69)

As students relate what they see on the screen (data) to what they know (warrants), inference possibilities emerge concerning concepts-in-action, and those inferences yield warrants, which in turn allow students to interpret data as evidence for or against a claim. What is particularly beautiful about these links is that they emphasize the students as initiators of their own development. The students, based on their immediate interpretations of the task and situation, generate data from which they can make inferences that lead to new realizations.

The insights provided by the links in paper 5 shed light on how the epistemic value changes through the use of a DGAE, as well as how students' processes can be seen as conceptually based mathematical inferences. By examining students' operational invariants and the inference possibilities between them, we can suggest how students exercise their RC when utilizing tools.

8.3.2 The three dimensions of the RC and mediation

Chapter 7 examines the three dimensions of RC (degree of coverage, radius of action and technical level) (Niss & Højgaard, 2019) and their connection to various aspects of IJ and the scheme-technique duality. Hence, it relates KOM to both the analytical tool and IAME's scheme-technique duality (Drijvers et al., 2013), including epistemic and pragmatic mediation.

This is illustrated in Figure 33, where elements that pertain to the degree of coverage are marked by a protractor, the radius of action by a compass, and the technical level by a calculator. In the figure, pragmatic mediation is represented by a yellow shade and epistemic mediation by a green shade.

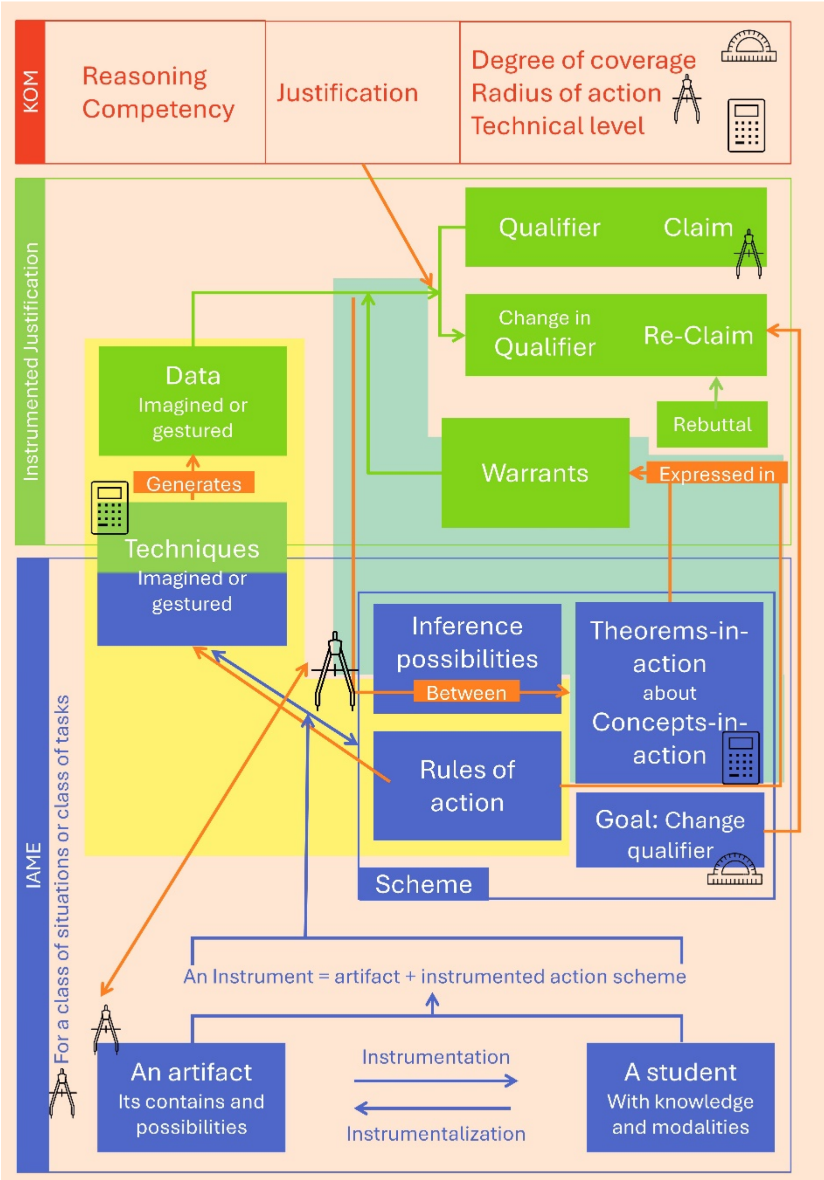


Figure 33 – Continuation of Figure 32. Added are the degree of coverage marked by a protractor, the radius of action marked by a compass, and the technical level marked by a calculator. In addition, pragmatic mediations are represented by a yellow shade and epistemic mediations by a green shade

The following section briefly summarizes how the three dimensions correspond to different aspects of the IJ tool.

Note that the degree of coverage is placed as related to the goal. This is the one link that relates most directly to KOM's RC, as the goal is expressed in students' active participation in IJ processes. Such an activity is aimed at changing the epistemic value of stated claims, distinct from other goals related to competencies, such as problem handling, mathematical modeling, etc.

The radius of action is expressed in several elements, some within the IJ process and some within the IAME. An element within the former is the nature of the claim being asserted, such as a proposed solution to a problem, a claim concerning a strategy, or techniques to solve it. The latter encompasses the range of artifacts a students can use for pragmatic and epistemic mediation in IJ processes across different mathematical domains. In Figure 33, pragmatic mediation is represented by a yellow shade and epistemic mediation by a green shade. This places the mediation in the processes that unfold between student's scheme-technique duality and the processes captured by the IJ tool. Pragmatic mediation occurs in the production of data through techniques that are governed by rules-of-action. Epistemic mediation occurs in the interpretation of data, as evidence by warrants, and in the relations between inference possibilities, the operational invariants, and warrants. Radius of action hence concerns both forms of mediation, in relation to the range of artifacts the students can use. Finally, it also concerns the students' range of tasks or situations for using artifacts.

The technical level is linked to the complexity of techniques used or envisioned. Nonetheless, the quality of justifications is primarily determined by a student's comprehension of underlying concepts and intrinsic properties, which are conveyed through warrants. The quality of justification is also influenced by the nature of the justification, i.e., whether it is grounded in phenomenological or algebraic knowledge.

8.4 DISCUSSION, PART 3: POTENTIALS FOR NETWOKRING AND REVISINTING NOTIONS

The discussion has two parts. 8.4.1 discusses meta-theoretical perspectives toward the potential of networking and networking strategies for linking KOM and IAME.

8.4.2 discusses the contributions of the links and, hence, revisits some of the notions introduced and elaborated in the theoretical framing in Chapter 2.

Finally, I discuss potentials for networking based on the theoretical development and the links established between the two frameworks as potentials for a coordinating strategy.

8.4.1 Potentials for networking

In networking practices (e.g., Bikner-Ahsbahs & Prediger, 2014), theories are described as close or distant to each other, depending on the extent of commonality. This section first addresses commonalities and differences of the two perspectives. This falls within the networking strategy of comparing and contrasting (see subsection 2.1.1). I draw on the concepts of *background theory* and *foreground theory* by Mason and Waywood (1996) and distinguish between *background theory* within or from outside of MER, as discussed in paper 6.

IAME is a theoretical approach that is local in grain size, and that theorizes students' use of tools in mathematics education. It has incorporated constructs from the theory of instrumental genesis (Rabardel, 1995/2002; Rabardel & Bourmaud, 2003) and developed constructs more particular to MER (Haspekian et al., 2023).

KOM is a medium grain size theoretical framework of hierarchically organized constructs. It deals with a broad range of topics related to the mastery of mathematics across different age groups and institutions (Niss & Jankvist, 2022). Thus, KOM and IAME differ in both grain size and scope. Since this study focuses on the RC, specifically on the aspect of justification, the theory development is local and aims to shed light on the processes involved in this aspect of RC.

"A fundamental issue for linking two theoretical entities is whether these represent two different ways of dealing with the same object(s) or phenomena, or whether they deal with different objects or phenomena" (Niss & Jankvist, 2022, p. 32). KOM and IAME have different objects, but they do share methodological means, as both concern student behavior as a subject of analysis. Consequently, the same datasets can be analyzed from both perspectives. Considering the complete frameworks of KOM and IAME, linking the two can be fruitful in specifying processes associated with a competency and relating them to different aims of tool use. This can enrich IAME with new perspectives on types of tasks and situations for tool use. In turn, for KOM, the competencies are extended to include aspects relating specifically to the use of tools. Linking the two approaches can thus be "mutually fertilizing" (Niss & Jankvist, 2022) for both approaches.

Both KOM and IAME concern the cognition of the individual. Competencies are considered cognitive in nature, since they are an individual's expression of cognition in specific mathematical situations (Niss & Højgaard, 2019). However, Niss and Højgaard (2019) clarify that *readiness* means "an individual's cognitive prerequisites for engaging in certain activities" (p. 18) does not have a distinct theoretical background theory outside of MER. Rather, it is developed as a framework inside of MER, from what appears to be a grounded theory approach (Vollstedt & Rezat, 2019), as the framework has emerged from an empirical discourse within a mathematical community, rather than resting on a scientific theoretical perspective. Instrumental genesis is a cognitive process that unfolds between an individual and an artifact, and IAME draws on the construct of schemes(Rabardel, 1995/2002; Vergnaud, 1998b). However, acknowledging a socio-cultural perspective with regard to artifacts (Rabardel, 1995/2002), IAME has a constructivist background theory (Mason & Waywood, 1996). In this regard, the two theories can complement each other.

Recall that networking can have different strategies depending on its aim and integration. Two of them are the coordinating and combining strategies (Prediger, Bikner-Ahsbahs, et al., 2008), which aims at understanding an empirical phenomenon or piece of data. What sets the two strategies apart is that combining considers conflicts between the theoretical approaches as a contribution to multifaceted insights into phenomena, whereas coordinating has different research objects. Therefore, research aims and methods must be complementary to capture inter-relational variables (Prediger, Bikner-Ahsbahs, et al., 2008). Taking the differences into account, coordinating is the most promising strategy. As the links established in this study aim at understanding the data that have emerged as instrumented justification, KOM and IAME also differ in the objects being studied, and the analytical tool for IJ suggests which inter-relational variables can be captured between the two.

Coordinating often results in a conceptual framework but does not necessarily present as completely coherent (Bikner-Ahsbahs & Prediger, 2014). However, coordinating aims at a high degree of coherency between well-fitting elements from different approaches. This leads to the question: How well-fitting is KOM and IAME?

Unlike IAME's notion of scheme, KOM categorizes activities and related cognition as particular to different situations, without capturing cognitive processes. This distinction, while posing the challenge of linking constructs at different grain sizes (Niss & Jankvist, 2022), concerning different objects, also highlights the complementary nature of the two perspectives. Hence, IAME and KOM do not have deviating constructs, which underscores the potential for their integration.

The present study addressed these challenges by adapting Toulmin's argumentation model. From a networking perspective, I will now argue that Toulmin's model, when reinterpreted into the analytical tool for IJ, can serve as a boundary object (see 2.1.1.5). This entity acts as a bridge between practices, making the boundaries between KOM's RC and IAME more permeable. The IJ analytical tool provides constructs that concretize the RC, which allows for the establishment of links. Star and Griesemer (1989) argued that a boundary object must be "plastic enough to adapt to local needs" and "yet robust enough to maintain a common identity" (p. 393). Both are relative matters. Concerning plasticity, Toulmin (2003) make few assumptions about mathematics or education. However, in the book *Uses of argument*, Toulmin (2003) does take a position on standards of argumentation. He is concerned with what counts as convincing for humans across scientific fields, and he is opposed to viewing argumentation by the standards of philosophies of logic. Though not completely aligned with mathematical ideals of arguments (that is, logic), this position makes the model adaptable across scientific fields.

In paper 2, we questioned the robustness of Toulmin's model: "These stretches seem to be leading rather far from the initial model, and we wonder how appropriate it might be to still refer to Toulmin's model at all, also considering a posteriori how we have sort of "substituted" elements from the IA to parts of the model." (p. 458). The primary concern revolves around the shift in the unit of analysis and its implications. While Toulmin's initial work did not intend to examine justification processes that lead to the argument, it is worth noting that these processes must at least include the components of the final argument. Additionally, these processes necessarily involve the individual's role in developing these components, and it is not uncommon in MER to use Toulmin's model in the context of reasoning processes (Simpson, 2015). To conclude, whether Toulmin's arguments model maintains its identity is debatable, but from a pragmatic point of view, the reinterpretation of the model into the analytical tool for IJ has proven to be a useful boundary object for understanding students' use of tools in justification processes. This has allowed links to be drawn across the RCs of KOM and IAME.

To summarize, comparing the two frameworks reveals potentials for networking KOM and IAME. They do not conflict concerning background theory or deviating constructs, and they share subjects of analysis. There are, however, some issues in linking the two perspectives that must be addressed. These issues concern the lack of correspondence between constructs, due two differences in objects and the scope of constructs. This study suggests that networking, at least on a local scale, can be assisted by a boundary object. Such a boundary object must concretize the given competency in play and be plastic enough to be linked to IAME.

8.4.2 Revisiting notions in the perspective of the analytical tool for IJ

The parallel analyses yielded limited new insights into the influence of epistemic mediation on the epistemic state of a mathematical claim. However, they did help establish connections between students' goals of instrument use and their competencies. The theoretical connections established through the analytical tool offer a more comprehensive understanding of how pragmatic and epistemic mediation are specific to IJ processes.

The processes of mediation can be further elucidated by considering the components of schemes. Pragmatic mediation occurs in the production of data through techniques governed by rules of action. It is important to note that the mediation of objects is facilitated by the artifact (Rabardel & Bourmaud, 2003). In line with IAME, the pragmatic mediation of objects using techniques is not solely determined by the student's own rules-of-action, as is the case with traditional pen-and-paper tasks (Vergnaud, 1998b). Pragmatic mediation is also influenced by the algorithms inherent in the artifact. Following Vergnaud's terminology, we may refer to them as *instrumented algorithms-of-action,* some of which align with mathematical theory.

Epistemic mediation plays a crucial role in how students interpret data as evidence through the use of warrants in the relationships between inference possibilities and operational invariants. This suggests that students are interpreting *instrumented algorithms-of-action* into their own theorems-in-action, advancing their conceptual knowledge.

Recall that the technique-scheme duality was suggested as a pragmatic means to solve a methodical problem. It was articulated by Drijvers et al. (2013) for analyzing small data sets (either in time or in a number of individuals) and because of the practical need to discuss instrument use is such cases from the perspective of IAME. Subsection 2.3.1 discusses the notion of techniques, to be considered within the background theory of IAME, as the primary perceptual and gestural activities that involve the mobilization and implementation of instrumented action schemes encompassing all gestures involved in student activity when using a digital tool (such as hand movements during expressions, observing an artifact or activity on objects mediated by the artifact, articulation of imagined activity). Paper 5 suggests that techniques are regulated by students' rules-of-action related to the artifact, as rules of action are "responsible for generating behavior based on situational variables" (Vergnaud, 1998b, p. 229). Revisiting the notion of techniques as the mobilization and implementation of rules-of-action can clarify how techniques both relate to and are distinct from schemes.

Another important discussion, the *relative* invariance of schemes, is rarely addressed in the IAME literature (Pittalis & Drijvers, 2023). Often, the definition of schemes emphasizes that schemes are invariant which is recognized in the invariant behavior. However, it does not address the new situations and tasks for which students have not developed instrumented action schemes, which is a typical situation in education. It is, however, recognized by Vergnaud (1998a):

> Nevertheless students are often faced with situations for which they do not have any scheme available. Therefore they have no other way but to call schemes in the neighborhood, to try to decompose and recombine them, in order to form new schemes, with or without the help of the teacher or other students. (p. 173)

Two conclusions can be drawn from this passage: First, students will decompose or recombine schemes, and second, this will probably occur with or without the teacher's involvement.

In paper 5, I suggest addressing such a situation by analyzing the components of scheme rather than looking for patterns across similar tasks. This can yield further insight into how students deal with an unknown type of task, such as the prediction tasks and the equal points task.

When such a situation is addressed, the notion of techniques becomes particularly relevant. As invariant patterned behavior has not yet been developed, we cannot search for it. Instead, we can investigate how students search for efficient rules of action as they decompose and recombine neighboring schemes. The case of Lev and Rio in paper 5 is a great example of this, as I argue that they possibly use rules-of-action related to variables as a placeholder, along with rules-of-action related to variables as a general number. In addition, some measures can be implemented. Design principle C suggests how to support students in this process, e.g., by directing student attention to an object in the algebra view with the task wording, or indicating the epistemic value of the claim as argued in discussion, part 1 (see section 6.5).

Techniques can be even more relevant if we understand them as rules-of-action that are not necessarily components of a stable, fully-formed scheme. They can also be rules-of-action of neighboring schemes that students are trying to apply in the situation, toward the development of an instrumented action scheme. Hence, techniques are the primary perceptual and gestural activities that involve the mobilization and implementation of rules-of-action in the development of instrumented action schemes.

These reflections and the theoretical developments of the study contribute to the IAME, as they expand the notion of schemes and its components as described by Vergnaud (1998b) and show how those components are relevant in the context of tool use, not only with respect to RC, but in general. Indeed, possibilities of inference and the operational invariants have particular roles in justification processes, illuminating the cognitive 'ingredients' that KOM does not explicate.

9 FINAL DISCUSSION AND CONCLUSION

The discussion parts 1, 2, and 3 have separately addressed answers and results for each of the research questions. The purpose of this final discussion is to address methodological choices, argue for the relevance of the results to MER, and discuss issues across the three research questions:

RQ1: In what ways can tasks be designed to encourage lower secondary students to exercise their reasoning competency when using a dynamic geometry and algebra environments in the case of justification focusing on variable as a general number?

RQ2: What are the relationships between lower secondary students' scheme-technique duality when solving tasks developed for RQ1 in a dynamic geometry and algebra environment and their exercise of reasoning competency as justification?

RQ3: Which theoretical links can be established between reasoning competency and the Instrumental approach to mathematics education from the theoretical developments of the study?

The main answer to RQ1 consists of the design principles A, B, and C, the microworld of variable points, and the associated tasks.

The main answer to RQ2 consists of the elaboration of the three dimensions of RC in students' IJ processes, and how they relate to the scheme-technique duality. This led to suggestions for how students' IJ processes can be supported in the classroom and through task design. An additional finding is the indication of a hybrid conception between continuous and discrete understandings of the variable in the prediction of variable behavior, made possible by the DGAE environment.

The main answer to RQ3 are the theoretical links between KOM and IAME, established through the analytical tool for IJ, as potentials for networking with a coordinating strategy. Building on these links, notions within the framework of the study are revisited.

This discussion is divided into four sections. 9.1 revisits the methodology, 9.2 discusses the quality and relevance of the answers, 9.3 debates issues and findings across the three research questions that are not yet addressed, and section 9.4 discuss perspectives for further research and implications for practice, and 9.5 concludes the dissertation, summarizing the answers to the research questions.

9.1 METHODOLOGY REVISITED

The overarching goal of the study was to investigate the potential of tools that integrate algebraic and graphic features, such as the algebra view of GeoGebra, for students to exercise their RC as a means for theoretical development that links the KOM framework to theories in MER.

As presented in Chapter 4, in pursuit of the overarching research aim, the study has taken the DR approach, as well as a networking perspective, as explained in Chapter 2. DR could have been deemed sufficient as it produces both empirical and theoretical results, so why the networking perspective? The reason is that the networking perspective allows reflective research of the theoretical developments at a deeper level than allowed within DR. Networking practices provide sustainability to the theoretical results because conceptual differences are confronted, and potential theoretical conflicts in the background theories, both outside and inside MER, and principles are identified so that discrepancies can be considered. Hence, the networking approach also contributes to the validity of the DR study. I will address this aspect in the coming discussion of methods and quality of results, but first, I will revisit DR in regard to competencies.

9.1.1 Revisiting design research as competency-specific

DR is inherently iterative, aiming to bridge theory and practice in educational research. Chapter 4 elaborates on how the researcher's role can be considered twofold, with a creative perspective and an analytical perspective. This section will discuss how the study can contribute to an understanding of *competency-specific DR studies* and address the explorative aspect of the DR processes in relation to the two perspectives.

Although competency-specific DR studies are not explicitly theorized within DR methodology, they can be considered analogous to topic-specific DR. Though engaging with mathematical content remains essential for exercising any competency (Niss & Højgaard, 2011), in competency-specific DR, the objective is to facilitate students' engagement with a competency, such as justification, creating opportunities to understand and develop this competency. Hence, in competency-specific DR, content serves as a means, while in topic-specific DR, it is the goal (Gravemeijer & Prediger, 2019). This distinction significantly impacts the design process. Gravemeijer and Prediger (2019) claim that the preparatory phase forms hypothetical learning trajectories for topics, conjectured on theories about a possible learning process, and a possible means of supporting that learning process and relevant or useful goals. However, in this study, the preparatory phase has focused on what inhibits or assists students' engagement in reasoning processes, rather than learning processes or learning goals. Thus, the answers to RQ1 elaborate on this query.

From the creative perspective, exploring justification processes within MER, especially using GeoGebra (paper 1 and subsection 6.1.2), necessitated an explorative approach to task construction.

Competency-specific DR allowed some freedom; the algebra view had to be in play, and it had to provide students with an opportunity to exercise RC. The conceptual aspects emerged from explorations rather than being predefined. This creative space enabled the exploration of GeoGebra's potential through varied tasks. Prior experience with GeoGebra, inspiration from coding, and feedback from peers, educators, and supervisors were integral to the creative process. Although this approach allowed creativity, starting from scratch impacted task quality, which meant that only some students engaged in justification. Further development suggestions are discussed in subsection 6.5.3 and 7.4. Reproducing or redesigning tasks from other studies could have lessened negative impact of the explorative approach. The competency-specific DR study in conjunction with the explorative approach also influenced the retrospective analyses. In the retrospective analysis, search for the potential tasks for students' engagement in justification (as the prediction tasks) and which tasks resulted in interesting justification that we could hope more students would engage in (as the equal points tasks). Topic-specific DR studies can also have an explorative approach, but naturally differ in the retrospective analysis to evaluate students' progressions within the topic.

By reflecting on Gravemeijer and Prediger's (2019) descriptions, competency-specific DR shares characteristics with topic-specific DR but differs in research aims, impacting both analytical and creative aspects. As such, this discussion can add to our understanding of what DR can be, as well as provide methodological warrants for answers for RQ1 that encompass the development of Design Principles.

9.1.2 Revisiting networking of theories

The NT perspective has influenced several aspects of the study, which I elaborate on subsequently. The main function of NT has been to elaborate and solidify the links between RC and the scheme-technique duality as a form of mutual fertilization (Niss & Jankvist, 2022) of KOM and IAME.

In all components of the study, the IAME (Drijvers et al., 2013; Trouche, 2005) provided the theoretical perspective on students' use of DGAE. The approach offers several constructs that revolve around the different processes of students' interaction with tools. As the project evolved, the technique-scheme duality became central for zooming in on students' use of artifacts rather than branching out to generalize across uses. As this duality has been heavily debated and critiqued (as laid out in subsection 2.6.1), it has required new perspectives on the notion of techniques, drawing on the background theory of IAME, examining the works of Rabardel and colleagues (Rabardel, 1995/2002, 2001; Rabardel & Bourmaud, 2003). In doing so, networking as a research practice (Bikner-Ahsbahs & Prediger, 2014) has been valuable, bringing awareness about the importance of the implicit assumptions of theories, both to understand the criticism surrounding the notion of techniques and how techniques can be approached to accommodate the matter. The study has established theoretical connections between RC and the IAME, based on the theoretical developments of RQ3, particularly through the analytical tool for IJ. This involved synthesizing research results to enhance the understanding of the link between RC and IAME. To examine the scheme-technique duality in the context of reasoning processes, Toulmin's (2003) argumentation model has been reinterpreted into a procedural model, labeled the analytical tool for IJ, that has acted as a boundary object (Star & Griesemer, 1989) between the two approaches. In these processes, networking has been paramount for choosing a set of constructs that did not conflict with either KOM or IAME.

9.2 QUALITY OF THE MAIN RESULTS

Schoenfeld (2007) argues that the quality of research must be judged by its trustworthiness, generality, and importance.

He presents five key dimensions of credibility, which are related to the persuasiveness of the assertions presented in a research study. These dimensions include descriptive and explanatory power, prediction and falsifiability, rigor and specificity, replicability, and triangulation. I will discuss trustworthiness, generality and importance with concern to both design results in subsection 9.2.1 and theoretical results in subsection 9.2.2.

9.2.1 Trustworthiness: Validity and reliability

Bakker and van Eerde (2015) propose that validity and reliability in DR studies can be addressed using the notions of internal and external validity. Internal validity, which refers to the quality of the data and the soundness of the reasoning that leads to the conclusions, plays a crucial role in the quality of research results. A technique that may be applied to improve the internal validity of a DR study is the use of data triangulation in the retrospective analysis (Bakker & van Eerde, 2015). The many sources of data (transcripts, video, screencast, written products) collected in the project allowed for this type of analysis.

The data have analyzed with two different methods. The established relationships between student's scheme-technique duality and dimensions of RC are based on the comparison of seven pairs of students solving the same task. The internal validity of these results is reliable as the argument are backed by a diversity between the seven pairs. However, these results could be strengthened by replicating the method in the context of other tasks. This could potentially also strengthen the descriptive elements (Prediger, 2019) of those relationships.

Other results stem from case-based analysis using the analytical tool for IJ (paper 2, 3 and 4, 5 with additional analysis in Chapter 6). Some stem from a comparative approach, comparing students' arguments across iteration 2 and 3 (paper 4). The findings that rest on analysis of a singular key-case (Thomas, 2011) of students' IJ processes have less internal validity and consequently, these findings serve only as proof of existence (Shoenfeld, 2007).

The validity of the analytical tool for IJ has been applied in the analysis of students' justification processes using tools. As the tool has been applied across cases with detailed description of both conceptualization and application, both the validity and reliability of the tool as an explanatory theory element can be warranted.

As explained in section 7.1, the IJ tool has an analytical unit from a claim to a reclaim. The analytical unit has generally been applicable across cases. However, I would like to point to a few issues. Some IJ processes have no reclaim, as in the case presented in the parallel analysis (subsection 8.1.1). In their processes, the students shift their attention to representational structures and do not return to the claim and the implications of the relationships they identified. It is obviously not uncommon for students to shift or lose focus. Such cases can still be analyzed but are unsuccessful justification processes, as the qualifier is not changed. Another issue to be careful about when identifying the unit of analysis in data, is that not all reclaims entail a change of epistemic value. This is the case if no new data have been produced or no new interpretations have been enunciated.

9.2.2 Descriptive and explanatory power

In DR, the implementation of interventions in a naturalistic setting leads to a vast number of dependent and independent variables, which can challenge the explanatory power (Collins et al., 2004). Thus, the descriptive power of DR requires the researcher to take particular care in describing both the setting of the intervention, the learners, the resources and supports applied in the setting, and the collaboration and role of practitioners that the intervention concerns. The DR processes are described in Chapters 5, 6, and 7. Chapter 5 details when and where the experiments took place and gives information about the participating students and the data collected from the experiments. Chapter 6 explores the design processes in-depth, covering both task construction and snapshots of the retrospective analysis.

Additionally, Chapter 7 and papers 2-5 offer examples of retrospective analysis. However, Schoenfeld (2007) also signifies that descriptive power is not veridical but should focus only on what is essential. It is reasonable reflect on whether the DR project has been overly elaborated. There are certain aspects, such as sources of inspiration and the reparative presentation of progression in design principles, that fall outside the essential description. However, the explanatory approach in the DR study and the lack of prior knowledge on the uses of the algebra view (paper 1) necessitated transparency and clarity of the DR processes—addressing aspects of rigor, specificity, and replicability. This necessity might somewhat reduce the descriptive power but ensures a comprehensive understanding of the design principles that have evolved.

The study is underpinned by the assumption that students' use of the DGAE is connected to their RC. This assumption is based on the understanding that variations in students' use of artifacts result in different justification processes and final arguments. These variations were identified through a comprehensive analysis of students' final arguments (paper 4), their use of techniques, as well as differences in their epistemic and pragmatic mediation (Chapter 7). Therefore, the relationship between the use of artifacts and students' RC is clearly demonstrated as descriptive elements in the data (Prediger, 2019).

With concern to the explanatory power of DR studies, Prediger (2019) states that if a relationship between descriptive elements can be derived, it increases the explanatory power of such elements by pointing to cause-effect relationships between phenomena or structures. Discussion Part 1, which addresses the design results, comes before the analytical results of Chapter 6 and the theoretical developments described in Chapter 7. Since explanatory theory elements in DR are closely related to theoretical development (Prediger, 2019), the explanatory elements are addressed in this final discussion.

The study has produced four explanatory theory elements. The first is the analysis of students' justification processes using the analytical tool for IJ, as in papers 2, 3, 5, and Chapter 7. These elements explain IJ processes as the production and interpretation of data toward the change of the qualifier of a claim, relying on the student's scheme-technique duality. The second explanatory element is the relationships established between students' scheme-technique duality in IJ processes and the three dimensions of RC. Hence, it explains students' IJ processes as expressions of their RC, related to different components of students' schemes. The third explanatory theory element is the links established from a networking perspective, explaining how the processes described by the IJ tool relate to concepts in IAME and the three dimensions of RC. The final explanatory theory element is the relationships to other competencies, identified in paper 4, namely the problem handling competency and the symbol and formalism competency.

9.2.3 Predictive power and falsification

Together, the explanatory theory elements add to the explanatory power of the study. What has gained less attention is predictive power. Predictions are about whether the results contain a predictive element and whether it is possible to falsify the results (Schoenfeld, 2007). The design principles are predictive (Prediger, 2019); however, the study has been less concerned with the extent to which certain solutions or forms of justification occur under the circumstances created in the experiments. Schoenfeld (2007) argues that "the more theoretical claims can be examined and tested by data, the more there is the potential for refinement and validation" (pp. 86-87). Falsification of the results of this study exceeds the limits of this PhD project but also falls back on the choice of relying on case-based analyses. As a result, explanatory power has gained priority over predictive power.

9.2.4 Generalizability and transferability of results

External validity refers to the generalizability and transferability of the results. The limits of the results can be nuanced into four types of generalizability: claimed, implied, potential, and warranted, depending on how substantiated the assertion of generality is (Schoenfeld, 2007). The discussions also touch upon future perspectives of the results, providing opportunities for discussing their importance. Schoenfeld (2007) argues that the relevance and importance of the results are crucial, as they determine whether the results can contribute to theory and practice in the field of mathematics education. This underscores the significance of the research in advancing the field.

A key aspect of DR is the development of theory elements transferrable to other contexts (Bakker, 2018). Therefore, the design study should consider elements that are both descriptive and explanatory (Prediger, 2019), as well as elements that go beyond the specific context. From a DR perspective, this reflects the quality of the results and provides insight into the relevance of the design rrinciples beyond the specific context of IJ.

The design principles have aspects that are transferable to task design for reasoning processes and other DGAE contexts, having both implied and potential generalizability. A finding that can be generalized towards task design for other reasoning processes is that the intrinsic properties should be known to students but in problems novel to students. This is relevant in reasoning contexts that concern the development of a student's degree of coverage or radius of action. If technical level and introducing new concepts or properties is emphasized, one can consider using familiar problems to balance out unfamiliar concepts.

The analytical tool for IJ and the definition of IJ can be generalized beyond the scope of the study. The relevance of the IJ tool, first and foremost, derives significance from contributing to KOM's RC (Niss & Højgaard, 2019) in students' use of the tool, especially by addressing a lesser researched area of justification. As argued across papers and in this kappa, the IAME concerns the use of artifacts in general but does not have notions that draw out the different processes, particularly for mathematics. Consequently, it has most often been applied to problem-solving. Hence, the analytical tools for IJ are a significant contribution to our understanding of the epistemic use of artifacts in reasoning processes. The analytical tool both brings a new perspective to a debated construct of techniques (see subsection 2.6.1) and suggests how the components of schemes (Vergnaud, 1998b, 2009) relate to the use of artifacts. Some of the findings that have emerged from the analysis bring new insights to IAME in general and not only in the context of justification. For example, paper 5 shows that students' personal experience of both the efficiency of rules-of-action and techniques advance their instrumental genesis.

The relations between students' scheme-technique duality and the three dimensions of RC are highly relevant for comprehending the role of DGAE in students' justification processes. However, most of the findings related to RQ2 are confined to that context. Some of the suggested relationships may have relevance for other competencies. For instance, the technical level surpasses the techniques utilized and pertains to how students utilize the tool for epistemic mediation.

The study focused on the productive side of RC within justification, and the analytical tool has been developed particularly within this context. Hence, its use within this context is warranted through the continued refinement and successful application in the study. To what extent can the IJ analytical tool be applied to the analytical aspect of the RC? This should be further explored, for example, in the context of students analyzing and critiquing other students' arguments, including their tool use. There are some obvious concerns. Though the processes will also be aimed towards changing the status of the qualifier of a claim, what is the role of DGAE in such a situation? Does the DGAE only act as context, or are data being produced or reproduced? A fundamental relationship in the analytical tool is the production of data through techniques, and that relationship must be present in the situation for the tool to be relevant.

A valid question to consider is: in what other situations can the analytical tool be useful? Its applicability beyond reasoning processes is uncertain, as it follows Toulmin's (2003) argumentation model. However, it can be applied in other contexts that involve reasoning and utilize DGAEs to produce data for justifications and explanations. For instance, it can be used to provide activities where the axiomatic structure usually becomes clear in the final result, but processes leading to a proof can take on different forms. It is possible that the IJ analytical tool can also help in understanding the role of tools in such processes, indicating its potential usefulness in that context.

Chapter 8 establishes and discusses theoretical links between the KOM's RC and IAME. A few of these may be generalizable toward other aspects of the RC and different competencies. A link that has already been implied to be general in part 3 of the discussion is that the goal component of students' schemes, in general, relates to students' exercise of competency. Some links could be yielding insights for investigating other contexts of tool use. For example, I draw a link between the possibilities of inference (Vergnaud, 1998b) and the development of the operational invariants expressed in students' warrants. Due to the theoretical perspective of justification, in an IJ context operational invariants play the role as warrants. This will differ if we consider operational invariants in the context of other competencies. For example, in the context of problem-handling competency, what are the role of operational invariants if they are not warrants? So in the context of other competencies, can we find similar patterns of inference possibility that evolve students' operational invariants?

9.3 RESULTS AND ISSUES ACROSS THE STUDY

First, the subsection 9.3.1 discus the assessment of RC in IJ processes, then 9.3.2 elaborate on how the study contributes to our understanding of the use of sliders in the algebra view.

9.3.1 Assessing students' exercise and progression of RC within IJ

This section discusses how the study can contribute to assessing students' exercise of RC, based on the collective results of the study.

As discussed in discussion part 2 (see section 7.4), IJ takes a student-centered approach that embraces students' reliance on empirical or phenomenological knowledge, and it advocates for an inclusive view of students' use of tools as an exercise of RC. The aim of the study is not to evaluate or assess students' RC but rather to understand the processes by which students form justifications in the context of a DGAE. However, there are elements that touch upon assessing students' RC within the context of IJ. These are explored in this section. These elements also serve to consider the further progression of students' RC. To assess students' exercise of RC within the aspect of IJ, the assessor can consider:

- Are the students' instrumented actions aimed at changing the epistemic value of a claim? A basic application of RC involves using a DGAE for verification without explanation, as discussed in subsection 7.3.3. Verification can be effective and adequate in problem-solving contexts, but to determine if the verification falls within RC, it must be utilized to change the epistemic value of a claim. This can be challenging to ascertain, as the aim becomes clearer when students elaborate explanations for their justifications.
- Are the students considering the intrinsic properties (Lithner, 2008) of the task in their justification? As discussed in paper 4, this is crucial for students' progression toward theoretical justifications. The intrinsic properties serve as a foundation for students to connect their phenomenological impressions to algebraic expressions when using a DGAE. This also relates to students' understanding of the properties. In paper 5 and section 7.4, I discuss this in relation to discrete and continuous understandings of variables in dynamic behavior. How do the students understand these properties? Furthermore, are students aware of how these properties are expressed in the tool and how they may differ from mathematical theory?
- What is the nature of students' warrants? The ongoing discussion in papers 2 – 5 and chapters 6, 7 and 8 concerns whether students' warrants are phenomenological or knowledge-based.

The latter is considered more advanced than the former, but a combination of both is also a possibility.

- What is the nature of students' final arguments? Paper 4 categorizes students' arguments as phenomenological, numeric, geometric, and algebraic. The specific task at hand determines which types of arguments are desirable.
- What form do students' arguments take? While the sophistication of students' arguments is related to technical dimensions in part 2 of the discussion, the study has not addressed the various forms of students' arguments. However, in the context of DGAEs, the transition from generalizing upon a few examples to collective subsets becomes relevant. Additionally, the aspect of form becomes more important as students' progress towards providing proof.

While theoretical and knowledge-based justifications are often deemed desirable, it is important to recognize the value of phenomenological justifications and their educational significance.

In the instances of phenomenological justification processes that I have observed, such as those of Isa and Em in paper 3, the phenomenological justifications related to intrinsic properties demonstrate students' engagement with algebraic structure and their ability to apply and adapt their mathematical knowledge. This supports Olive et al. (2010), who argue that observing properties of invariance while manipulating the object has the potential to connect experimental and theoretical mathematics (p. 150). The more tangible experiences students have with algebraic expressions, the stronger their foundation is for further progress. Embracing these phenomenological experiences, which a DGAE can provide, as the basis for algebraic understanding may be an educational ideal for lower secondary school.

9.3.2 Uses of the algebra view and sliders for RC

Paper 1 describes the lack of literature about the use of the algebra view in GeoGebra together with sliders for lower secondary students. The paper concludes:

> Very little has been researched about which functionalities in GeoGebra's Algebra View for working with variables as a general number, as well as how to use the functionalities in task design for activating lower secondary students' mathematical reasoning competency. Still, the review does indicate that using the slider for explicit variables can be used for this aim, and typing in expressions containing variables should be further explored in the context of GeoGebra. (p.61)

While the potential for connecting symbolic representations to graphic representations is mentioned, this potential is unexplored within MER. The study advances the research within this area, i.e. students' use of DGAE. The main critique of the algebra view is its complex representational systems. This suggests the need for low-complexity task design, allowing students to capitalize on the potential. This need can to some extent be met by using explicit rather than implicit variables when students are asked to transform and construct objects through algebraic expressions. The microworld of variable points (see section 6.2) is a contribution to understanding how the low-complexity of task design can be realized. Indeed, the microworld of variable points offers novel representations of algebraic properties. By constructing points and dragging sliders, students can access various phenomenological experiences of the structural properties of algebraic expressions. Additive structures can be experienced as positions of sets or trajectories in the coordinate system. Multiplicative relationships can be experienced as differences in the length of the sets, or as the speed points move with, using the animation feature. It also provides phenomenological experiences of intrinsic properties related to algebra and the variable as a generalized number, such as limits, infinity, and equality.

The slider plays a crucial role in both pragmatic and epistemic mediation in IJ. Students' utilization of it can be categorized into production of data for either verification or justification. Students' RC can be linked to the sophistication of their justifications in relation to a technique, rather than the complexity of the technique itself. This underscores the significance of using the slider tool for data generation and interpretation, as a significant tool in the algebraic view for reasoning processes.

Through the DR study, the results stemming from the design provide insight on how the use of the slider links graphic representation and symbols, by providing access to phenomenological experiences of algebra concepts and structures. In the context of RC, ideally, it is such a link that students attempt to justify. However, for students to progress from phenomenological justification to knowledge-based justification, support and encouragement through task design and teacher guidance are necessary. In addition, as discussion parts 1 and 2 argue, phenomenological experiences depend on students' utilization of the graphic view, capitalizing on the potential for linking it to the algebra view.

9.4 IMPLICATIONS FOR PRACTICE

Some of the results have particular implications for practice, which I would like to emphasize here.

The potentials, and in particular potential uses, of technology are an ongoing debate. However, there is some common ground for considering the epistemic uses valuable. What this study brings to this discussion is that epistemic use of digital tools in reasoning processes involve capitalizing on students' phenomenological experiences as a foundation for mathematical reasoning in justification.

Some design results are relevant for implementation, both in the form they take in this kappa and in the papers, but also as inspiration for uses of GeoGebra in practitioners own design processes. In this context, it is an advantage that the microworld exists within a piece of software that is already known and accessible to students and teachers, at least in Denmark.

By providing a new representational structure, the microworld of variable points can be applied by practitioners for teaching and learning algebraic concepts, both as a microworld for RC as well as other competencies and age groups. This study is directly relevant to lower secondary school and basic algebraic concepts and properties. However, more complex algebraic relationships, such as exponential and quadratic relationships, can also be explored in the microworld.

The predictions tasks are an example of the task structure “Justified Prediction-Observation-Explanation tasks” (see subsection 6.1.1) with particular importance for practice. It can involve students of all ages and in different institutions when using a DGAE tool - also beyond the RC. The “Justified Prediction-Observation-Explanation” tasks have already proved valuable in reasoning processes using DGEs (Højsted, 2021). This study adds to that knowledge by supplying details of how such a task can capitalize on the different phases, for example, to predict within the environment by drawing, tracing, or dragging free objects. It has also been shown that comparison of other objects in the prediction can assist students in justifying differences in the objects. Hence, the study provides examples of how predictions are valuable in students’ exercise of RC.

The three-part discussions and this concluding discussion have collectively laid the foundation for the subsequent project conclusion. The final section emphasizes the project’s contributions to the overarching goal and the research inquiries.

9.5 CONCLUSION

This dissertation has explored the potential of using a DGAE, which integrates algebraic and graphic features, for students in lower secondary school (aged 13–16) to exercise their RC in justification processes. Additionally, these investigations have contributed to theoretical development by linking RC in the KOM framework with the use of DGAE to theories in MER.

Together with the kappa, I have reported on the study in six individual, yet related, papers, which have contributed to addressing the three research questions through theoretical and empirical contributions.

The research questions have been addressed with the iterative processes of DR, including a networking perspective. Networking has played a dual role as a platform for reflecting on the theoretical decisions in the project, as well as linking the KOM to other theories in MER. Though the results of the kappa may seem segregated between theoretical and empirical, I will, once again, stress that both empirical and theoretical results are obtained through the iterative processes of DR, as explained and illustrated in chapter 4.

The study has addressed how tasks can be designed to encourage lower secondary students to exercise their RC, particularly in the context of justifications that focus on variables as general numbers (RQ1). The main results establish three design principles. The first principle deals with the balance between unknown problems and students' knowledge of concepts in reasoning contexts. The second principle focuses on the complex representational structures of DGAE. The third principle suggests a prediction task structure for promoting reasoning processes. These design principles have been applied to and refined through the construction and development of tasks that encourage students' exercise of RC using a DGAE in a microworld of variable points.

Furthermore, the study has examined the relationships between students' *scheme-technique duality* (Drijvers et al., 2013) and their exercise of RC when solving these specifically designed tasks (RQ2). The examination has included an analytical tool for IJ, presented in the appurtenant papers (2,3 and 5). The results describe RC in relation to the use of DGAE in three dimensions: degree of coverage, radius of action, and technical level

Finally, the study has established theoretical links between RC and the IAME, drawing on the theoretical developments of the study (RQ3), that is, in particular, the analytical tool for IJ. This has involved integrating insights from the research to contribute to a deeper understanding of how RC and IAME can be linked. This theoretical perspective of the study has helped elaborate on RC and the scheme-technique duality in a form of mutual fertilization (Niss & Jankvist, 2022), and it provides the potential for the networking strategy of coordinating (Prediger et al., 2008). The later stages of the study focused on Vergnaud's (1998b) notion of scheme, which served to elaborate on conceptual aspects of linking IAME to the RC in the KOM framework.

What are the potentials of using DGAEs to help students (aged 13–16) exercise their RC in justification? The use of DGAEs not only serves as an entry point for students to explore fundamental algebraic structures and concepts, but also as a vital tool in the process. The study underscores the need for several support measures from the side of both task design and teachers. However, if these measures are carefully thought out and implemented, the link between algebraic symbolism and dynamic graphic representation, through the use of sliders, can provide phenomenological experiences of algebraic concepts and structures, allowing students to engage in reasoning about other abstract concepts.

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Paper 1

Gregersen, R. M. (2022). How about that algebra view in geogebra? A review on how task design may support algebraic reasoning in lower secondary school. In U. T. Jankvist, R. Elicer, A. Clark-Wilson, H.-G. Weigand, & M. Thomsen (Eds.), *Proceedings of the 15th International Conference on Technology in Mathematics Teaching (ICTMT 15)* (pp. 55–62). Danish School of Education. https://ebooks.au.dk/aul/catalog/book/452

HOW ABOUT THAT ALGEBRA VIEW IN GEOGEBRA? A REVIEW ON HOW TASK DESIGN MAY SUPPORT ALGEBRAIC REASONING IN LOWER SECONDARY SCHOOL

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This paper reviews the existing literature for insights on how functionalities in the Algebra View in GeoGebra can be used in task design to activate lower secondary school students' reasoning competency when working with variables. Through an extensive review, only a small number of studies were identified, indicating that this area of research so far has been neglected. Nevertheless, the studies included point towards the slider tool to be useful, as it allows students to test their conjectures about the mathematical relationship a variable represents and to experience if-and-only-if statements. More specifically, typing in algebraic expressions containing variables in an input field can orient students' reasoning towards the symbolic representations.

Keywords: Algebra, GeoGebra, reasoning competency, task design, variable as a general number.

INTRODUCTION

In this paper, I consider the potentials of GeoGebra's Algebra View to support students' reasoning with expressions and variables as a generalized number by reviewing existing literature in the field of mathematics education, with a focus on lower secondary mathematics education.

Within the last decade, the use of digital technologies in mathematics education has increased. In Denmark, this development has coexisted with the implementation of the idea of mathematical competencies (Niss & Højgaard, 2019) in the Danish mathematics programs, and both developments have been supported by the educational policies (UVM, 2019). This to such an extent that the interplay of digital tools and students' mathematical competencies have become the subject of research (Geraniou & Jankvist, 2019), as new didactical potentials and possibilities emerge when new technological tools are introduced in educational practices (Artigue, 2002).

In Denmark, Dynamic Geometry Systems (DGS), and in particular GeoGebra, have been implemented in mathematics education throughout primary and lower secondary education. GeoGebra holds the common features of a DGS but also differs by incorporating algebra, geometry, and calculus in the same dynamic software (Hohenwarter et al., 2009). What can be considered unique for GeoGebra is the so-called Algebra View (Wassie & Zergaw, 2018). All graphical objects are simultaneously expressed algebraically and numerically in this panel. Through an input field in this panel, objects can be constructed. This includes geometrical objects, but also algebraic objects such as variables (expressed by a slider), functions, groups, etc. It has functionalities such as measuring, counting objects, logical and boolean conditions, as well as functionalities similar to Computer Algebra Systems (CAS). Yet, it also holds functionalities that go far beyond, and the syntax is considerably different. That the Algebra View provides the option for students to work algebraically with mathematical objects seems in line with the developments of early algebra. Already in 1998, Kaput (1998) pleaded to algebra school mathematics across all ages, leading to an increasing number of studies and projects focusing on younger students' early algebraization (e.g., Cai & Knuth, 2011). Yet, little research has focused on the development of algebraic reasoning in lower secondary school (Knuth et al., 2011). Variable in school mathematics is used as a symbol of an 'unknown quantity', as a ‘general number’ for any indeterminate quantities or, in functional relationships as ‘covariation’. Generally, the research on the potentials of the use of DGS in mathematics teaching and learning has focused on conceptual development in Euclidean geometry as well as simple and complex functions (Wassie & Zergaw, 2018), including also the bridging of these two mathematical domains (Pedersen et al., 2021). Consequently, the research largely focused on either co-variance in functional relationships or invariants in geometric constructions. More specifically, the research focusing on the potential for DGS to support students’ reasoning and reasoning competency has largely focused on the teaching and learning of Euclidian geometry (e.g., Højsted, 2020). How DGS and GeoGebra specifically can be used to activate students’ reasoning competency when they are working with variables as a general number is less researched. In this paper, I attempt to draw out of the literature what has been researched in this matter. Hence, I ask *What existing literature for which functionalities in GeoGebra’s Algebra View can be used in task design for activating lower secondary school students’ mathematical reasoning competency when working with variables as a general number?*

THE KOM-FRAMEWORK AND ITS REASONING COMPETENCY

The Danish mathematics competency framework (KOM) (Niss & Højgaard, 2019) was initially developed for educational use and as such describes what mathematics as a disciple demands of cognitive processes in terms of competencies. One of the outsets for this endeavor was to overcome the understanding of school mathematics as only concerning the learning of the subject matter, but also to encompass a set of competencies that reflect what is distinctive for mathematics practice in the society. The framework constitutes eight competencies. KOM defines a mathematical competency as “someone’s insightful readiness to act appropriately in response to a specific sort of mathematical challenge in given situations” (Niss & Højgaard, 2019, p. 6). In this review, I will focus on the reasoning competency. The reasoning competency is associated with situations where students analyze or produce mathematical arguments. These can be oral or written arguments and in a range of justifications form from exemplifying to deductive and formal proof. An argument is considered to be a chain of statements linked by inference to justify mathematical claims or solutions to mathematical problems. To consider what are appropriate actions in a particular situation is also restrained by the mathematical topic and the problem posed. At the lower secondary school level, the curricular goals for the competency are that students should be able to distinguish between individual cases and generalizations, as well as develop and evaluate mathematical reasoning, including when working with digital tools (UVM, 2019). The reasoning that goes on in the classroom at this level is, however, informal, and only a few—or maybe even no—deductive proofs are dealt with in class. Yet, it is at this stage that students are expected to be able to put forward justifications of mathematical relations that to a higher degree rely on theoretical knowledge and, to a lesser extent, their intuition.

METHOD

The literature search was done in five stages. In stage one, relevant texts were found by database searches in ProQuest and Web of Science, limited to the educational databases and texts in English.

In ProQuest (Hits=115) the following search string was used (10th August, 2020):

- noft(GeoGebra) AND noft(Algebra* OR vari*) AND la.exact("English" OR "Danish") AND la.exact("ENG") NOT edlevel.exact("Higher Education" OR "Postsecondary Education" OR "Adult Education") AND PEER(yes),

In Web of Science (Hits=50), two sets were created and combined (10th August, 2020):

- Set #1 (GeoGebra) AND noft(Algebra* OR vari*)

- Set #2 NOT ("Higher Education" OR "Postsecondary Education" OR "Adult Education")

20 duplicates were found using *Jabref.* All in all, 146 texts were identified. Also, the following conference proceedings were screened by searching for "GeoGebra" and then identifying any use of the Algebra View in the identified papers. This was done for CERME (N=8), ICTMT 10th- 14th (N=2), MEDA (n=0). All 156 texts were uploaded to *Covidence*, where another five duplicates were identified. The remaining 151 texts were then abstract-screened, and 48 were full text screened, following the inclusion/exclusion criteria seen in table 1. Studies that used other software in a manner that was highly similar to that of the Algebra View have also been included in the review.

	Inclusion criteria	**Exclusion criteria**
Tool use (1)	Use of GeoGebra or highly similar software, where algebra features are explicitly used.	Only use of geometric features. Not GeoGebra or software with highly similar features
Age group of participants (2)	Primary and lower secondary.	Kindergarten, adult students, in-service and preservice teachers, university or college students
Types of students (3)		Students with dyscalculia, deaf students, and students with special needs
Types of studies (4)	Empirical or theoretical.	Studies without any documentation, or any description of or reflections about students' interaction with algebra functionalities
Mathematical Content (5)	Variable as a general number.	Co-variance/functions, statistics, programming, STEM

Table 1. Inclusion and exclusion criteria

Stage 1:	156 references imported for screening
	5 duplicates removed
Stage 2:	151 studies screened against title and abstract
	103 studies excluded
Stage 3:	48 studies assessed for full-text eligibility
	44 studies excluded: 16 (criteria 1); 9 (criteria 2); 10 (criteria 4); 7(criteria 5); 2 (not English or Danish)
Stage 4:	4 studies included
	1 study included from sources identified in reference list
Stage 5:	In all 5 studies included

Table 2. Prisma of inclusion/exclusion process

By snowballing references, "Future curricular trends in school algebra and geometry: Proceedings of a conference" was identified as a source, and after the screening, one study was added.

PRESENTATION OF STUDIES AND FUNCTIONALITIES IN THE STUDIES

The five identified studies are all peer-reviewed but cannot be perceived at the same quality of a journal paper. This indicates that the research of the potentials of GeoGebra's Algebra View, and its functionalities for mathematical tasks and processes other than functions, is still in a developing phase. Two of the papers are theoretical, while three present empirical results. Four of the five studies make use of GeoGebra, and one study by Lagrange and Psycharis (2011) makes use of a programming "turtle world" software, LOGO, which makes use of very similar affordances to that of the Algebra View in GeoGebra. The software makes use of programming language, whereas the Algebra View in GeoGebra uses standard algebra notations and commands specific to the program.

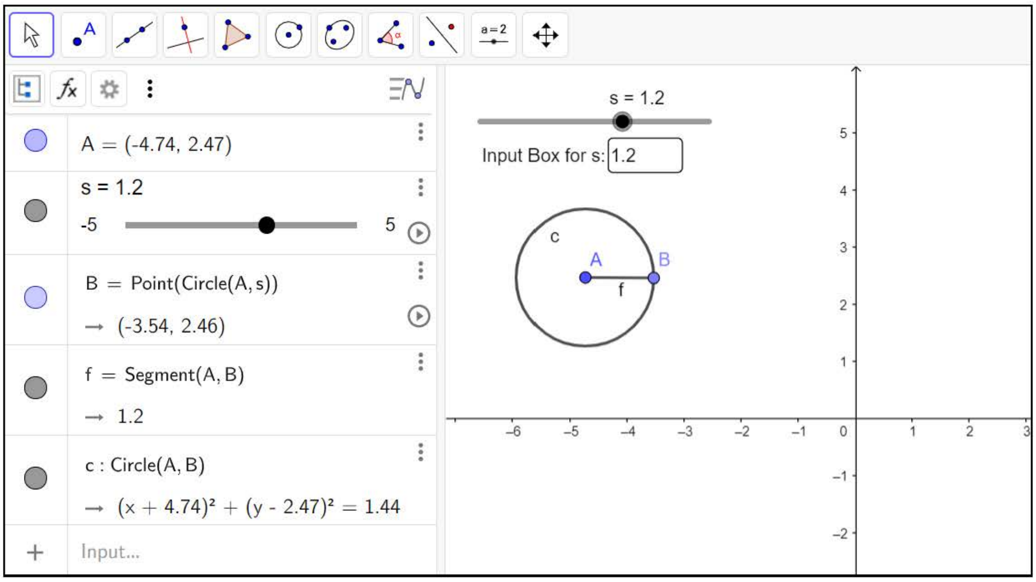

Figure 1. GeoGebra Classic online. A slider along with an input box, varying the radius of the circle *c* and line segment *f*. All visible both on the Graphics View and the Algebra View

The functionalities investigated in the five papers are the slider (in some cases also controlled by an input box) and typing expressions containing one or more variables in the input field. In the analysis of the studies, I distinguish between variables that appear explicitly and implicitly in the Algebra View. Please, refer to figure 1 for the following descriptions of functionalities. The explicit appearance is the slider tool is created by typing in the name or the letter of the variable in the input line, which produces a line segment with a dot on it (none of which are actual geometrical objects), a numerical value, and the numerical limit of the variable, which by default is -5 to +5. These limits are changeable. When dragging along the line segment, the numerical value changes accordingly. One can also adjust the increment by which the numerical value changes. A slider is visible in the Algebra View, and can also be displayed in the Graphics View. A slider can control any geometrical object in the Graphics View by defining the object by a variable. For example, in figure 1, the radius of circle *c* is defined by *s*, and hence also the length of segment *f* =*AB*, which is why both can be varied by the slider. The implicit appearance of variables happens through the construction of any dynamic geometrical object on the Graphics View, which then appears in the Algebra View with a name, definition, and value. For example, a line segment, as in figure 1, will appear with a name, e.g.,

f (a variable), defined by its endpoints, AB, and its value in terms of length, which in this case can be varied by dragging the slider as point B is restricted to the circle.

ANALYSIS OF STUDIES

The two theoretical studies, Mackrell (2011) and Jawkin (2010), are critical towards the integration of geometry and algebra in GeoGebra. Jawkin (2010) does acknowledge that analytical geometry is an obvious connection between these two domains and agrees with the intentions of the software to tackle student resistance towards algebra. One major concern, however, is that much of analytical algebra is not within the scope of elementary school mathematics. Furthermore, he argues that the unification of algebra and geometry in GeoGebra poses a pedagogical issue in the infrastructure of representation. For example, what appears to be a circle in the Graphics View is actually a plotting of a quadratic function, which loses its circular shape if coordinate axes are changed. Mackrell (2011) experiences that the construction of geometrical objects in the Graphic View that produces implicit variables can result in discrepancies in the representations in the Algebra View. For example, if a circle is dragged, the equation for the circle varies, which is an algebraic representation, whereas if a segment is dragged, then it is the measurement of its length that varies, which is not an algebraic representation (notice f and c in figure 1). Mackrell (2011) exemplifies the difficulties of using the Algebra View to calculate the relationship between the area of the circle and the radius, not only because of the discrepancy in algebraic representations but because of the large amount of information. Jawkin (2010), on the other hand, defends this discrepancy by considering the complexity of the algebraic representation that students would have to face if a line segment was represented by its equation and limits. Despite Mackrell (2011) being critical, she points out that the slider "has the potential to be an important link between geometric and algebraic representations" (p. 3), but she does not investigate the use of the slider any further. Mackrell (2011) and Jawkin (2010) both point out issues that must be considered when developing task design for GeoGebra, both in general and when designing tasks that aim at activating students' reasoning competency when working with variables. Interestingly, none of the three empirical studies seems to encounter the issues that are described here, which might be explained by the fact that they all use variables explicitly through the slider and not implicitly.

The first example of explicit use of variables is that of Lagrange and Psycharis (2011). In the task posed, they ask students to dilate an alphabet letter proportionally dependent on a single variable controlled by a slider, using LOGO. They describe how a slider provides a linkage between the algebraic and the geometrical representation by "providing a link between the graphical distortion and the symbolic aspect" (p. 199). They argue that by dragging a slider, the status of the *physical system* is connected to the status of the *symbolic system*. This allows students to conjecture about cause and effect between the numerical values and the visual variants depicted in the Graphics View. In the study by Lagrange and Psycharis (2011), the students' only possibility to change the physical system is through the symbolic system. This supports that students' reasoning about the relation between the symbolic and physical system is oriented towards the symbolic system, since students must produce conjectures about algebraic expressions and test them by dragging the slider. This explicit use of variables and students' possibility to algebraically act in the system indicates that it is possible to direct students' reasoning toward algebraic expressions.

Using sliders to validate or refute conjectures about the relations between numeric values and geometrics relationships is also elaborated by Soldano and Arzarello (2017). The task design in this study only partly uses functionalities of the Algebra View. The Algebra View is hidden, but a slider and input boxes are depicted in the Graphics View. Their task is a so-called "Hinitikka Semantical game", where students must compete while investigating under which numerical circumstances two circles tangent. They do so by manipulating three sliders controlling the radius of each circle and the distance between the circles. The Graphics View also depicts the numeral value of three variables. The students must discover that these express the absolute value of the difference between the radii, the sum of the radii, and the distance between the centers of two circles. Soldano and Arzarello (2017) find that by dragging sliders to represent different generic states in the configurations in the Graphics View, or to consider different values of a variable, students challenge each other's claims by producing examples and counterexamples, leading students to find and even appreciate 'if-and-only-if' relationship. The task design supports students to reason about which geometrical relationships the variables resemble through two different uses of the sliders. Nevertheless, the students' argumentations are not oriented towards algebraic expression, as we saw in Lagrange and Psycharis (2011). Possibly this is because the students cannot test algebraic expressions in the task.

In Tanguay et al. (2013), the use of variables and sliders is oriented towards reasoning about arithmetical relationships. They conduct an apriori analysis of a task design that displays two sliders along with input boxes in the Graphics View, hiding the Algebra View as in Soldano and Arzarello (2017). One slider controls the n-number of isosceles triangles grouped around the center, and the second slider controls the angles in the center. Students are to identify cases of when an n-sided polygon is formed, which is when the sum of the angels in the center is 360 degrees. In the case that students calculate the angle, they can type it into the input box. The students are thus brought to examine, within a geometrical context, the list of divisors of 360. The instances (e.g., $n = 7$) that do not form an n-sided polygon approximation are then to be explored by increasing the number of decimals of the center angle, leading the students to experience rational numbers and the decimal limits of GeoGebra. Here the input box is utilized as the increment of the slider becomes very sensitive for a large number of decimals. We see the use of variables and sliders as a means of exploring and reasoning about arithmetical entities on geometric representations. Again, similar to Lagrange and Psycharis (2011), the students cannot test algebraic expressions.

DISCUSSION

Considering the two appearances of variables, the explicit use of variables is dominant in all three empirical studies, whereas the implicit use of variables is discussed in the two theoretical papers. Also, two out of the three empirical studies hide the Algebra View, and the study that does not hide it uses LOGO and not GeoGebra. The tool that we gain the most insight into is the slider, in two cases along with an input box and in one case along with the possibility to type in expressions. Several points can be drawn considering task design for activation of students' mathematical reasoning competency. I will synthesize these in the following.

To design tasks that support the students' activation of their reasoning competency, the slider provides a link between the graphic representations, the algebraic representations and the numeric values. Dragging the slider represents the variation of a numeric value, which allows students to test conjectures about the mathematical relationship the variable influence or represent by testing and receiving feedback from the system. This can either be for different values of the variable(s) or different states of the objects depicted in the Graphics View. In Tanguay et al. (2013) and Soldano and Arzarello (2017), we see that using the sliders only in the Graphics View can support students to activate their reasoning competency, as they can explore mathematical relationships, form conjectures, and possibly experience 'if-and-only-if" relationships. This can be related to the "cause and effect" of dragging the slider. However, leaving the variable as singular entities on the Graphics View without giving access to the Algebra View limits the students' possibilities to test their conjectures as algebraic expressions. Task designers and mathematics educators must be aware that there is a potential in GeoGebra for students to test conjectures about algebraic expressions, thus allowing students to engage further into reasoning with variables. In Lagrange and Psycharis (2011), we see students who engage in reasoning about algebraic relationships by testing if the typed algebraic expressions result in a successful dilation of a letter and who reflect upon the results.

In the two studies, Tanguay et al. (2013) and Soldano and Arzarello (2017), input boxes are used along with the slider, allowing the students to easily test specific values of the variable, and this without struggling with positioning the slider. If there is no access to the Algebra View, task designers should be mindful of this possibility as it allows students to test exact values more easily.

Despite GeoGebra being, at least in Denmark, one of the most used DGS in primary and lower secondary school, the review reveals that there is a surprisingly small amount of studies that make use of the functionalities of GeoGebra's Algebra View in task designs that use variables as a general number. This makes one wonder if the Algebra View simply is too complex for younger students to manage. It is clear from Mackrell (2011) and Jawkin (2010) that the algebraic representations do impose challenges, not least in terms of discrepancies in the algebraic representations and the amount of information assessable in the Algebra View. Task designers and educators should keep these issues in mind when designing tasks using functionalities of the Algebra View. As in Tanguay et al. (2013) and Soldano and Arzarello (2017), designers can hide the Algebra View altogether, but there are also other possibilities to limit accessible information in the Algebra View. For example, pre-constructed objects can be hidden, or the Algebra View can be set to only show descriptions or values, making the information less complex. In addition, using the explicit appearance of variables instead of the implicit appearance of variables can ease these issues. Nevertheless, there is still much to be discovered about how functionalities in the Algebra View can be used in task design for activation of students' reasoning competency when it comes to variables as a general number. In Lagrange and Psycharis (2011), we do get indications of how typing in expressions that contain a variable that is simultaneously influenced by graphic representation in LOGO can do exactly this. Will a similar design bring similar results if tried out in GeoGebra? And how can we design tasks drawing on these functionalities to engage lower secondary students in mathematical reasoning on core concepts and structures in algebra, such as generality, equality, additive, and multiplicative structures?

CONCLUSION

What can be concluded from this review is that, in general, very little has been researched about which functionalities in GeoGebra's Algebra View for working with variables as a general number, as well as how to use the functionalities in task design for activating lower secondary students' mathematical reasoning competency. Still, the review does indicate that using the slider for explicit variables can be used for this aim, and typing in expressions containing variables should be further explored in the context of GeoGebra.

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Paper 2

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Developing an analytical tool of the processes of justificational mediation

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Within the Instrumental Approach (IA) the newly developed notion of justificational mediation (JM) describes mediations that aim at establishing truth of mathematical statements in the context of CAS-assisted proofs in textbooks. Here we study JM with the intent to broaden the notion to the context of informal justification processes of early secondary students interacting with GeoGebra. Seeing JM as a process that has the objective of changing the status of a claim, we use Toulmin's model and combine it with the IA to unravel the structure of the process through an analytical tool. The study is part of a broader project on the interplay between reasoning competency and GeoGebra with lower secondary students.

Keywords: digital environment, Instrumental Approach, justificational mediation, reasoning competency, Toulmin's model.

REASONING COMPETENCY AND JUSTIFICATIONAL MEDIATION

During the last decades, the use of digital technologies in mathematics education has increased, as well as the body of research in this area (e.g., Hoyles & Lagrange, 2010). In Denmark, this development has coincided with the promotion of *mathematical competencies*, seen in the KOM-framework as "…someone's insightful readiness to act appropriately in response to a specific sort of mathematical challenge in given situations." (Niss & Højgaard, 2019, p. 6). In the wake of this development, a need has arisen for understanding the interplay of students' enactment and development of the specific mathematical competencies and their use of digital technology (Geraniou & Jankvist, 2019). What might "readiness to act appropriately" mean in the context of digital technology? How can such readiness be identified and nurtured? These are examples of broad questions that gave rise to this study.

We follow Geraniou and Jankvist (2019) who took some first steps in weaving together the KOM framework with the Instrumental Approach (IA), which is also widely used in the European research community. The IA suggests that the use of tools involves *pragmatic mediation,* concerning the subject's actions on objects and *epistemic mediation*, concerning how the subject gains knowledge of objects' properties through the tool (Rabardel & Bourmaud, 2003). However, Jankvist and Misfeldt (2019) suggest that a third form of mediation, *justificational mediation* (JM), may be useful in the context of CAS in proofs and proving activities. JM concerns how the status (e.g. probable, likely, true or false) of statements for a student is modified through the use of a digital environment (Jankvist & Misfeldt, 2018; 2019). However, the authors have advanced the notion of JM within the context of CAS-assisted proofs in textbooks in upper secondary school, which touches on the more formal part of the *reasoning competency*. Still the authors ponder whether students think about justification, insight and performing mathematical labor as different things and how (Jankvist & Misfeldt, 2019). So, other situations relating to the less formal side of the reasoning competency spectrum should be considered and studied separately within this frame.

Within the KOM-framework's reasoning competency, we study students' mathematical informal argumentations that take place within the digital environment GeoGebra, focussing on the processes through which an uttered statement changes status: it may either be rejected or believed to be true to a greater degree than in its initial form. The ways in which students justify their claims within an environment like GeoGebra can assume forms that are closely related to the environment itself, as well as to the underlying mathematical theory within which the objects are placed. Hence we ask: *how can we analyze JM and what insight into it can we gain*?

Seeing JM as a process of argumentation, our analytical tool is derived from Toulmin's model, and, because JM occurs in a digital environment, we make use of constructs from the IA. We now explain how the theoretical frame is set up.

THEORETICAL FRAMEWORK: CONSTRUCTING A TOOL OF ANALYSIS

Although the original intention of Toulmin's model was to analyze finalized argumentations (Toulmin, 2003), there are numerous examples in mathematics education where it is used to analyze students' processes of argumentation (e.g. Pedemonte, 2008; Simpson, 2015), also in the context of digital environments (eg. Hollebrands, Conner & Smith, 2010). These studies, however, do not usually situate the model within the research field of educational use of digital technologies in mathematics, and hence do not draw on the theories used in this field. In this study, we suggest an analytic tool that does exactly that.

With respect to the IA, we consider GeoGebra as an *instrument*. Such a notion arises from the use of an *artefact* and the development of *scheme*. In this context the artefact is GeoGebra itself, but in other cases it could be a specific tool within it (such as dragging, or a slider). *Schemes of utilization* are developed by a solver to accomplish a specific task (Rabardel, 2002). *Scheme* is understood according to Vergnaud's construct: "the invariant organization of activity for a certain class of situations" (Vergnaud, 2009, p. 88), that relates an "invisible part" to a student's visible actions. Schemes are made up of various aspects, including a *generative aspect:* rules to generate activity; namely the sequences of actions; information gathering; and controls and an *epistemic aspect:* operational invariant; namely concepts-in-action; and theorems-in-action, with the function to pick up and select the relevant information and infer from it goals and rules.

In the following, we will introduce elements from Toulmin's model and explain how we interpret them within the IA and with respect to JM.

JM through Toulmin's model in the context of GeoGebra

In Toulmin's model, the *claim* is a statement of the speaker, uttered with a certain indication of likelihood (*qualifier*); the claim is justified through other elements of the argument (data, warrant, backing). The first utterance of the claim indicates the start of the JM process, in which the *aim* is to change the qualifier. In younger students' informal argumentation, the aim is seldom to construct rigorous mathematical proofs but rather to convince themselves of the existence of mathematical relations and facts (Jeanotte & Kieran, 2017). Hence, a change in the status of the qualifier will often be from likely to more likely, and less often from likely to true. We recognize such a change of status of a claim by students' restatement of the claim accompanied by a new qualifier. The change of the status is reached through the generation of *data* that for the solver constitutes evidence and facts supporting the claim, and through the *warrant* that consists of inference rules that allow the solver to connect the generated data to the claim (Toulmin, 2003). The warrant is often implicit, in which case, it must be inferred from the utterances and gestures of the students. We can infer the warrants and analyze the generation of data through the notion of scheme introduced above. The *generation of data* is the product of the generative aspect of the schemes used (e.g., dragging, creating objects on the screen and interacting with them, utterances and other hand-gestures) that are carried out by students. Warrants are the epistemic aspect of the schemes used. One last element remains; *backing*. This element requires some careful consideration, which we elaborate in the next section.

Toulmin describes the *backing* of a warrant as "…other assurances, without which the warrants themselves would possess neither authority nor currency" (Toulmin, 2003, p. 96). However, Simpson (2015) identifies three different uses of backing in mathematics education research. In the context of JM, we consider the backing to be an explanation of why the warrant is relevant (Simpson, 2015). Central is, that the aim of JM is to change the status of the claim, so the backing must explain why the warrant is relevant for generating data that allows the change in the status of the claim. Thus, the backing becomes fundamental to the JM process. Currently, we have reached the following formulation of *backing* in JM processes:

If the claim is true, I can generate data, within the specific instrument, that is consistent with the claim.

This seems closely related to Vergnaud's (2009) notion of *theorem-in-action*, a sentence that the solver believes to be true, but that may in fact be false. Though it can be, it is not a mathematical theorem, and it can bridge domains of different natures. In our case it bridges the phenomenological domain of GeoGebra with the theoretical domain of algebra (also see Baccaglini-Frank, 2019). We recognize, that there might be variations of such a formulation, but we are currently studying this form.

METHODOLOGY

The task we analyze in this paper comes from a broader project, in which a series of tasks were designed by the first author and assigned to students in three classrooms of grade 7 students (in all 61 students). All students had prior experience using GeoGebra's geometric tools, as well as constructing points and sliders in the algebra view, but they had never used the slider to vary points, which is central in this task. The students worked in pairs for two 90-minute sessions while being video recorded. All together 17 pairs was recorded. The video recordings captured the screens and the students, both from the computer's camera and from a second handheld camera controlled by the first author, who was present during all the sessions.

The example below, is of a pair students, Lilly and Mia, who were described by their teacher as a particularly "talkative" pair, who usually participated with confidence to math class, even though they were not considered to be "the best" students. The tasks was posed and solved in Danish. The task as well as the excerpt have been translated to english for this paper.

We selected this example because of its short length and the fact that it contains many aspects of the process of JM. Indeed, in these 75 seconds the students changed the status of an initial claim from likely to more likely. This episode, therefore, constitutes a unit of analysis.

AN EXAMPLE AND ANALYSIS OF JM

We use the following transcript to illustrate the analytical tool and how it is applied in an analysis of students' justification processes. The two students are working on a task, where they are asked to predict how two given points $A = (1,s)$ and $B = (s,1)$ will move in the coordinate plane in GeoGebra. If the two points are constructed in the algebraic view, a slider for the interval [-5,5] will appear for the variable s, the slider can either be dragged or animated, and its movement induces the points to move in the coordinate plane as s varies. To ensure that the students predict, rather than construct and animate/drag the slider, the GeoGebra interface in this specific task is limited to the graphics view, showing a coordinate plane along with the cursor, the point tool, and the pen tool. An orange textbox also appears with the coordinates of the given points.

Lilly and Mia make a conjecture about a *line* through AB and discuss it, despite the task does not mention any lines. Lilly holds the mouse throughout the excerpt.

1.	Lilly:	[Reads out the task] Show in the coordinate system how you think point A and B move as s changes value.
2.	Mia:	I have the feeling they are making such a slanted line like this (Fig. 1a).
3.	Lilly:	Yes.
4.	Mia:	That is what I imagine.

5. Lilly: If I make a point now called A right? [Places point with point tool in (2, 2.98)] So this, this is A.

6. Lilly: And then we can say that ehm, A is equal to one comma s, right? [moves the curser to point at the coordinate sets in the orange text box (Fig. 1b)]

7. Mia: What should s be?

8. Lilly: One here, and then s could be… [Moves A towards (1,0)]

9. Mia: Four.

10. Lilly: Four, so it will be here then [Moves A to (1,4) (Fig. 1b) along x=1]

11. Mia: Yes.

12. Lilly: Then we do B.

13. Mia: [Points to approx. (4,1) with her index finger (Fig. 1c)] Yes, that is what I said, then it becomes such a slanted line.

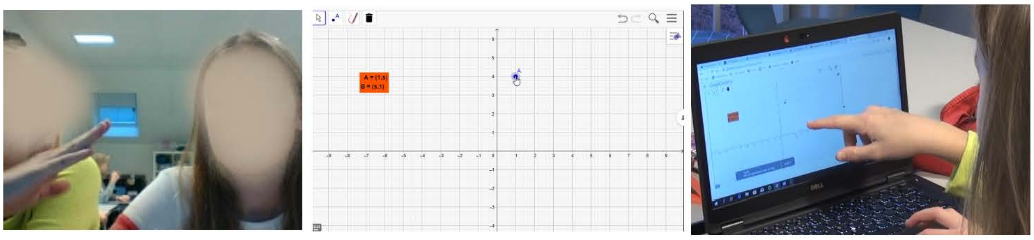

Figure 1. a, b, and c:
a) Mia's gesture, b) screenshot of Lilly's placement of point A, c) Mia points to screen approximately at (4,1)

Analysis of the example

In the analysis, we identify the structural elements and relate them to JM. Figure 2 on the next page visually illustrates Lilly's and Mia's JM process.

A process of JM starts in Lines 2-4 when the following claim (C_1) is stated and gestured: "they [A and B] are making such a slanted line like this" along with the qualifier "feeling" which indicates likelihood, not certainty. Lilly seems to base her claim on the initial data consisting of the algebraic expressions $A = (1,s)$ and $B = (s,1)$; moreover, she describes the line in her claim through a gesture (Fig. 1a), identifying certain geometrical features of such a line, possibly its "slant". Now the students go on to generate data for the claim using the instrument with the aim of changing the status of the claim, as we are about to show.

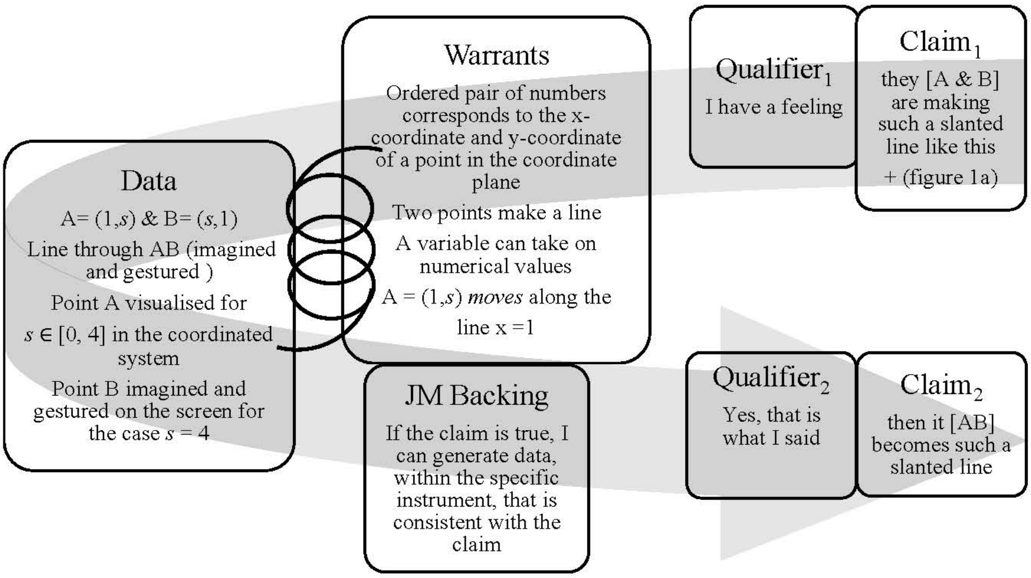

Figure 2: Illustration of Mia's and Lilly's JM process

Throughout lines 5-11 new data is generated by using the instrument. In line 5 and 6, the students create a point and establish the relationship between the algebraic notation of A and the created point. Throughout lines 7-11 data on this relationship is expressed by moving the point on $(1,s)$ from (1,0) to (1,4). On this basis we infer the warrants, schemes, used by the students to connect the data to the claim: an ordered pair of real numbers corresponds to the x-coordinate and y-coordinate of a point in the coordinate plane; two points makes a line; a variable can take on any real number; and $A = (1,s)$ moves along the line $x = 1$. We note that the third warrant depends on the instrument, as the *movement* of points only exists tacit within the instrument. This is an example of how warrants can contain both theoretical elements and phenomenological elements, linking the algebraic domain to the GeoGebra environment, as we discussed earlier. We infer the backing to be what we conjectured: If the claim is true, I can generate data, dependent on the specific, instrument that is consistent with the claim.

In lines 12 and 13, the students generate data regarding point B that is imagined and gestured on account of the same warrant and backing as lines 5-11. In addition, the restatement of the claim in line 13 indicates a change in its status of the claim: the utterance "Yes, that is what I said" suggests that the qualifier has changed from likely to more likely. Overall, to reach the change in status the students drew on their conceptual knowledge, as well as their knowledge about how variables are expressed within the tool. The restatement of the claim and change in its status also concludes a unit of analysis for the process of JM.

CONCLUDING DISCUSSION OF OUR ANALYSIS

In this study we seek to gain insight into informal argumentation processes as part of what we consider the students the reasoning competency and how this interplay with their use of GeoGebra. We do this with a focus on a particular form of mediation - justificational mediation (Jankvist & Misfeldt, 2019), arising from research that combines the KOM-framework and the IA (Geraniou & Jankvist, 2019). Here we designed an analytical tool inspired by Toulmin's model and grounded within the IA. In the following we will discuss and reflect upon the insights we have gained of this endeavour.

The use of Toulmin's argumentation model has allowed us to identify and amplify the importance of the qualifier as indication of change of status of a claim. This has served as a structure for identifying a unit of analysis of what can be considered a processes of JM. This supports that such a mediation is governed by the aim of changing this status of a claim; it has also allowed us to connect the generative aspects and epistemic aspects of schemes (Vergnaud, 2009) to the structure of an argument.

However, there are also limitations with this approach that relate to Toulmin's argumentation model. We do not yet find that this tool appropriately captures the crux of the matter, which is the interplay between theoretical and phenomenological components in students' informal argumentations. Aspects of this interplay can be seen through the notions of scheme and theorem-in-action, that we have adapted to the warrants and backing of the model. This adaptation feels like a long "stretch" with respect to what Toulmin's model has been previously used for in mathematics education (Simpson, 2015). Moreover, we have transformed Toulmin's model into a structure with two claims (or rather a first claim and then its restatement) and two qualifiers, to highlight the process of change in status of the claim and how it occurs. These stretches seem to be leading rather far from the initial model, and we wonder how appropriate it might be to still refer to Toulmin's model at all, also considering a posteriori how we have sort of "substituted" elements from the IA to parts of the model. Moreover, we have not yet been able to explicitly interweave the KOM-framework with the theoretical lenses used. To sum up, has referring to the IA and to Toulmin's argumentation model together supported us in understanding JM? To some extent yes, as it has provided some insight into students' instrumented activity involved in changing the status of a claim; however, it does not yet completely satisfy us.

ACKNOWLEDGMENTS

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Paper 3

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Lower Secondary Students' Reasoning Competency in a Digital Environment: The Case of Instrumented Justification

Rikke Maagaard Gregersen and Anna Baccaglini-Frank

1 How Does the Use of Digital Technology Influence Students' Mathematical Reasoning Competency?

The work presented in this chapter is part of a broader research problem stemming from the following Danish education context that, however, is arguably an important matter in other countries. In Denmark, the Mathematical Competencies framework (KOM) (Niss & Højgaard, 2019) highly influences the curricular goals (UVM, 2019). KOM defines a mathematical competency as "…someone's insightful readiness to act appropriately in response to a specific sort of mathematical challenge in given situations" (Niss & Højgaard, 2019, p. 6). In all, there are eight distinct yet interrelated competencies; here, we will be focusing in particular on the *reasoning competency*.

Although KOM, which was developed at the start of the century, acknowledges digital technology in mathematical practices, it does not account for the prevalence and the role that digital technologies now play in mathematics programs at all educational levels. In Demark, GeoGebra is the primary dynamic geometry environment (DGE) used early on for mathematics teaching (Højsted, 2020b). Indeed, DGEs are considered to support students' mathematical reasoning competency (e.g., Højsted, 2020a). For example, they can support students in connecting mathematical theory with empirical explorations or identifying geometrical invariants as key properties of geometrical figures and relationships (e.g., Højsted, 2020c; Leung et al., 2013; Sinclair & Robutti, 2013).

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Our work stems from the need to deepen digital technology aspects of KOM's competencies descriptions, as Geraniou and Jankvist (2019, 2020) advocated at a practical and theoretical level. Specifically, we intend to contribute to partially bridging this research gap by offering a theoretical tool to analyze how a digital interactive environment like GeoGebra can contribute to lower secondary school students' reasoning competency in a mathematical domain at the crossroads between algebra and geometry.

This contribution is also quite relevant from an internationally broader perspective. Indeed, in addition to being a DGE, GeoGebra features an "algebra view" with the symbolic representations of items that appear in the graphic view. This is a feature shared by computer algebra systems (CAS) in general that has been studied especially in the context of functions and related concepts in calculus (e.g., Artigue, 2002; Drijvers et al., 2013; Lagrange, 2010, 2014; Takači et al., 2015). However, the potential of dynamic geometry and algebra environments is yet to be fully unveiled (Hohenwarter & Jones, 2007), especially at the lower secondary school level.

In the following paragraphs of this section, we will clarify what is intended in the KOM framework by *reasoning competency* and how we intend to approach it. Then we will provide an overview of our conceptual framework, explaining how we adopted each construct, connecting it with others, to reach the theoretical tool that we designed by putting it in relation to Toulmin's argumentation model (from now on Toulmin's model) and the Theory of Instrumental Genesis. We will then use the tool designed to study students' argumentation processes in an interactive digital environment; specifically, we analyze excerpts from two students' efforts at solving a task in GeoGebra in which the objects in play are described algebraically and graphically. Finally, we will discuss our findings, leading to the notion of *instrumented justification* to frame the process captured by the analytic tool.

1.1 Reasoning Competency in the KOM Framework

The reasoning competency includes the ability:

- to produce oral or written arguments (i.e., chains of statements linked by inferences) and to justify mathematical claims;
- to critically analyze and assess existing or proposed claims and justification attempts.

So the competency explicitly considers *justification*, hinting at various forms of justification, ranging from reviewing or providing examples to rigorous proof (Niss & Højgaard, 2011, 2019). Niss and Højgaard (2019) also note that reasoning goes beyond justifying theorems and formulae, extending to the justification of any mathematical conclusion obtained through mathematical methods or inference.

This way of describing reasoning competency—especially the first ability presented—resonates highly with research on *argumentation*. Indeed, we situate our work within this discourse, and we use "argumentation" to refer to all processes aimed at producing and validating mathematical claims.

Many studies have shown that students can struggle with both identifying the relevant properties and structuring a mathematical argument (e.g., Duval, 2007). Moreover, when engaging in argumentation, students might rely on authorities such as standard formulas, teacher's statements, or technology instead of their mathematical knowledge (Harel & Sowder, 2007; Lithner, 2008). Argumentation commonly aims to change the epistemic value of a mathematical claim (Duval, 2007). Consistently with Jeannotte and Kieran (2017), we consider justification a specific type of argumentation process "… that, by searching for data, warrant, and backing, allows for modifying the epistemic value of a narrative." (Jeannotte & Kieran, 2017, p. 12).

1.2 Designing an Analytical Tool from a Complex Theoretical Panorama

Our work is situated in a rather complex conceptual framework that we need to clarify, explaining mutual relationships between the theoretical approaches and the theoretical constructs we use. As discussed above, the broad framework within which we situate this work in the KOM is a quite general framework organizing the main competencies needed to become a proficient mathematician. However, it lacks detail for students' uses of digital technology. The compatibility of KOM with a theory designed specifically to analyze students' use of digital technology has already been explored by Geraniou and Jankvist (2019). Using the same theories, we take a step into further analytic detail to gain insight into students' reasoning processes, specifically justification seen as a particular process of argumentation supported by digital technology. To do this, we use Toulmin's model, designed to capture the structure of argumentations and adapt it to the context of a digital interactive environments using the *scheme-technique* duality from the Theory of Instrumental Genesis (TIG). We do this with the intention to understand the empirical phenomenon of students' justification processes in a digital environment.

The analytical tool we introduce here is re-elaborated from the one presented in Gregersen and Baccaglini-Frank (2020). The TIG describes how an artifact such as GeoGebra can become an instrument for an individual who engages in solving a task (Rabardel & Bourmaud, 2003). Moreover, Drijvers et al. (2013) have elaborated within the TIG three dualistic processes; we consider the *scheme-technique* duality. Techniques are considered the "visible part" of doing that relies on the "invisible part", the solver's schemes, that direct and organize techniques. Moreover, schemes contain concepts and rules which regulate actions. This duality assumes that part of the scheme can be inferred from observing actions.

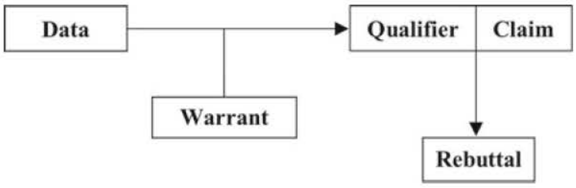

Fig. 1 Basic elements in Toulmin's argumentation model (Toulmin, 2003)

Traditionally, the TIG has been applied to gain insight into students' learning processes solving specific mathematical tasks using a digital environment, for example, finding the solutions of an equation with CAS (e.g., Artigue, 2002; Jupri et al., 2016). In our case, students will be using GeoGebra to solve mathematical tasks, but they might also be arguing in favor of or against certain claims arising in their solution process, and we are interested in capturing this. The scheme-technique duality alone is not sufficient, as we want to gain insight into students' justification processes, as particular argumentation processes, so key structural aspects should not get lost.

We, therefore, use Toulmin's model (Toulmin, 2003) to keep track of these processes. In Toulmin's model, the *claim* is a statement of the speaker, uttered with a certain indication of likelihood called the *qualifier*. This is supported by *data* that is facts and *warrants* that are inference rules which connect the data to the claim. Finally, the *rebuttal* denotes conditions for or limits of the claim. Figure 1 depicts such a model, as introduced by Toulmin (2003).

Commonly for Toulmin's model, the unit of analysis is a finalized argument restricted to a single sentence. However, a key aspect of the justification processes we aim at capturing is the change of the qualifier of a claim, possibly leading to the rejection or restatement of the original claim. Therefore, our units of analysis consist of students' actions (including utterances and gestures, both technology-mediated and not) between their first utterance of a claim and a restatement of the claim, that we call *re-claim,* involving a change in the qualifier. The qualifier can then be inferred from the student's actions; for example, a statement can be uttered with hesitation, or if a student continues to search for data, we can infer that the student is not yet convinced that the claim is true. The qualifier can change from "possible" to "more possible", "less possible", "true", or "false". To change the claim's qualifier, the students argue in favor or against the initial claim as they generate *data* that constitute factual evidence. Figure 2 shows a generic diagram of our adapted Toulmin's model, our new analytical tool: in the top right corner, noted in gray, is the first uttered claim along with a qualifier; below is the re-claim, with a new qualifier.

A second feature of our analytical tool is that a *technique* frame appears next to the data. This is because the main source of data, as students attempt to justify claims in a digital interactive environment like GeoGebra, is the effect of their use of techniques (as described in the TIG). The invisible schemes direct and organize actions with or on the data, but they also contain conceptual elements and rules that regulate actions (Drijvers et al., 2013). Such rules can be seen in the model as *warrants,* which are inference rules that connect the data to the claim.

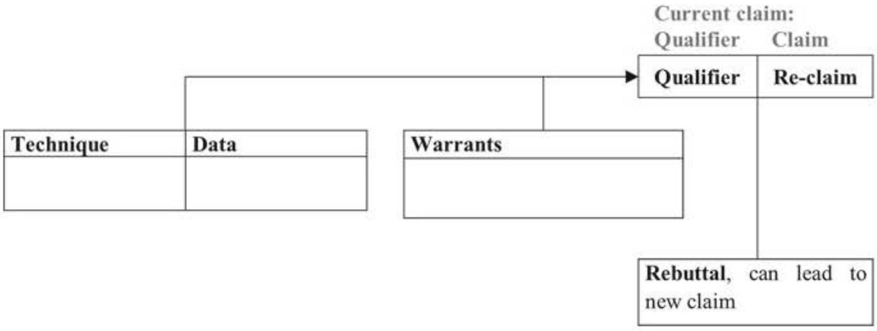

Fig. 2 Adaptation of Toulmin's model: an analytical tool for students' justification processes in a digital interactive environment

In short, we look at actions as warrants connecting data to claims: we make inferences on the students' (usually implicit) warrants, through their verbal utterances and visible actions, to gain insight into their justification processes and, more in general, into their reasoning competency. For example, since warrants consist of inference rules connecting data to the claim, they direct how data is interpreted as evidence (Baccaglini-Frank, 2019). Making such inferences about students' warrants, thanks to the structural setup provided by the tool, seems to be completely coherent with Toulmin's (2003) statement about warrants being potentially implicit.

A justification process can be made up of various justification sub-processes (Pedemonte, 2007). Each of the sub-processes constituting units of analysis can be analyzed using the tool in Fig. 2. Since a justification sub-processes can build on previous sub-processes, for the same student or pair of students (see the following section), analyses of successive sub-processes through the analytical tool may contain long lists of data and warrants. For this chapter, we do not graphically link successive justification sub-processes, but we take into account previous units by recalling relevant data and warrants previously generated and used in the new unit analyzed.

2 Method

In the case introduced below, the task that the students are solving is taken from a sequence of tasks designed by the first author. It stems from her doctoral work, with the general aim to explore the potentials of basic tools in the algebra view of GeoGebra (e.g., typed-in expressions, sliders, variable points) concerning students' justification processes. The sequence of tasks was assigned to students in pairs in three 7th grade classrooms in two 90-min sessions. All students had prior experience using GeoGebra. 17 pairs agreed to be recorded as they worked on the tasks, capturing their screens, faces, and voices to make more accurate inferences, especially about the qualifiers of their claims. The transcriptions in this chapter have been translated from Danish to English.

The students were asked to work in pairs so that through the interaction with a peer, we could gain more insight into the students' justification processes. This approach is common in the use of Toulmin's model in mathematics education research (eg., Fukawa-Connelly & Silverman, 2015; Knipping, 2008; Pedemonte, 2007). Acknowledging that Toulmin's model originally only takes into account a single individual's argumentation, we keep track of discrepancies between each student's position with respect to their warrants by labeling warrants that seem to be held by one (and not the other) student. If students seem to hold the same warrant, we do not label it.

2.1 Task Design

In the task we consider in this chapter, students are given the points A $= (1, s)$, and B $= (s, 1)$ (see Fig. 3) and asked to construct a point C, dependent on s, so that C and A move in parallel [directions]. Then they are asked: *Can C = B? If so, when?*

The algebra view and its tools are accessible, but the toolbar is restricted to the cursor, the line construction tool, the parallel line construction tool, and the perpendicular line construction tool (see Fig. 3). This design choice was made to ensure that the students used the tools accessible in the algebra view. Previous tasks introduce the trace tool to create a trace mark of dynamic points dependent on the variable by dragging the slider. The slider can also be animated to make the variable change "on its own". This was not introduced, but it was used by some students, including those presented in the case here. The default range of a variable represented

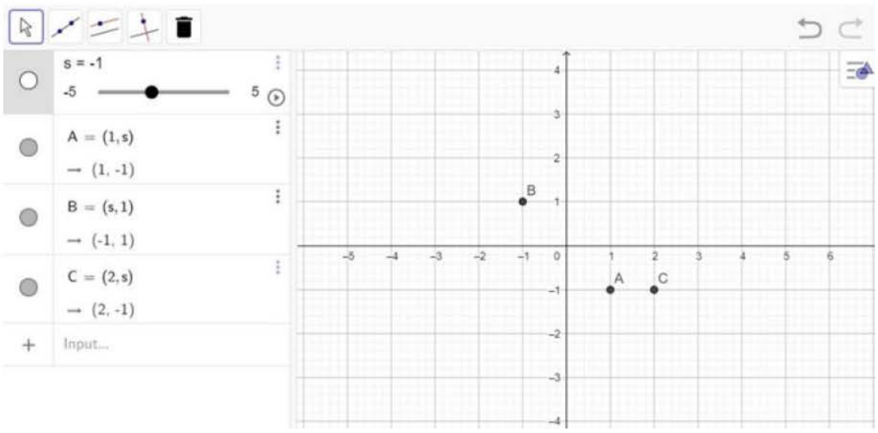

Fig. 3 GeoGebra setting of the task. To the left is the algebra view. From top: a slider for the variable s, points A, B, and a possible construction of C, all dependent on s and displayed in the graphics view on the right with the coordinate plane

by a slider in GeoGebra is $[-5, 5]$; it can be changed, but for this task, it was left to this default setting.

We now provide a preliminary analysis of the task concerning possibilities it offers for the students' justification processes. For our students' age group, we only expected justifications less formal than proof.

The mathematical concepts at play are variable points on the coordinate plane, equality, parallelism, and intersection of lines (or segments). How these concepts are represented in GeoGebra's graphic view increases the complexity. Although the task refers to "parallel movement", students who see this mathematically as points belonging to parallel lines may have a deeper insight into how to solve the task. We will clarify how the mathematical concepts can come into play in the solution of the task.

The lines, and therefore parallelism and intersection, are indirectly represented. One of the coordinates of each point is defined by a variable, so the points move on the plane as the variable is changed. The coordinates of each point do not just refer to a single point on the plane but to a set of points restricted by the limits of the variable ($[(-5), 5]$ as set by default), describing a segment that is either parallel or perpendicular to the x-axis. These segments can be represented by activating the trace of the variable point describing it when the slider is dragged. If the slider is animated (i.e., it moves automatically), it provides the opportunity of focusing on the movement of the dependent objects, in this case, points A, B, and C. With the coordinates given, points A and B have the same coordinates when $s = 1$, but B and C cannot be equal, that is, they cannot occupy the same place on the screen at the same time. However, the position and movement of A, B, and C can be altered by changing the expression containing s of their coordinates, either by adding a term or changing the coefficient. The latter also changes the length (and hence the set of points) described by the trace mark.

Depending on the students' knowledge of generality, they may approach the task "Can C = B? If so, when?" in different ways. If they only consider the *specific point* C that they construct, C = B only if their point C intersects with B in a single point, with fixed coordinates, that hence need to be identical for both B and C. If, instead, the students consider C as a *set of points on a specific line parallel to the trace of A*, for example, $x = 2$, the point of the intersection of the traces left by C and B identifies a possible equality. If the students consider C as the *set of all points on any line parallel to the trace of A*, the answer could be a general expression such as "C = B, if C = $(d, \frac{1}{d}s)$" or C = $(d, s - (d - 1))$, when $s = d$. Of course, this is beyond our expectations for the students in this study.

2.2 Presentation of the Case

We selected episodes from the work of two 14-year-old girls, Em and Isa, because they were one of the two pairs of students in the larger study who answered that it is possible to have B = C. Figure 4 presents the GeoGebra applet, with the points and

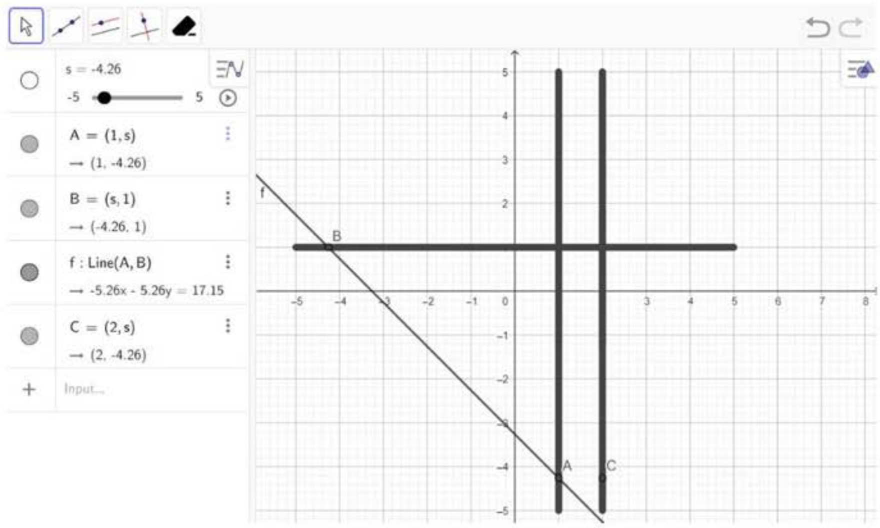

Fig. 4 Screenshot of Em and Isa's screen showing trace marks from A, B, and C. Replication: https://www.geogebra.org/m/khjnewpm

line constructed as they started the task. It displays: points A = $(1, s)$ and B = $(s, 1)$, along with point C = $(2, s)$, constructed by the students, and a line connecting B and A, which is unrelated to the task at hand. The trace is active for all three points, and the animations are turned on for the slider of s.

3 Analysis of the Students' Justification Process

In all excerpts, Isa is controlling the computer. The first author acted as a teacher-interviewer in the classroom along with the regular math teacher.

In the transcription, "what is done" and authors' notes are enclosed in square brackets. The transcript is presented in three excerpts that capture the main justification sub-processes of their general justification process. After each excerpt, we provide its analysis through our analytical tool. We label reappearing warrants in bold: the label **WE** concerns Warrants of Equality of points, **WP** concerns position and behavior of points, and **WT** concerns Traces.

Excerpt 1: justification sub-process 1

1	Isa	Okay then, can C be equal to B?
2	Both	[Observe GeoGebra until line 10].
3	Em	Collide, collide, collide—no
4	Isa	No, they cannot be equal to each other. Because these [points at A then B with cursor], they can be equal to each other.
5	Isa	But C can't.
6	Em	They will never collide.
7	Isa	That is because C, C is too slow.
8	Em	It is too far away.
9	Isa	Yes.
10	Em	It is too far away.
11	Isa	At least the C we have made cannot.
12	Isa	[Types answer: "no it cannot"].
13	Isa	Okay, justify your answer [Reads from written assignment].
14	Em	But there will probably be some that can if they are further away (Fig. 5).

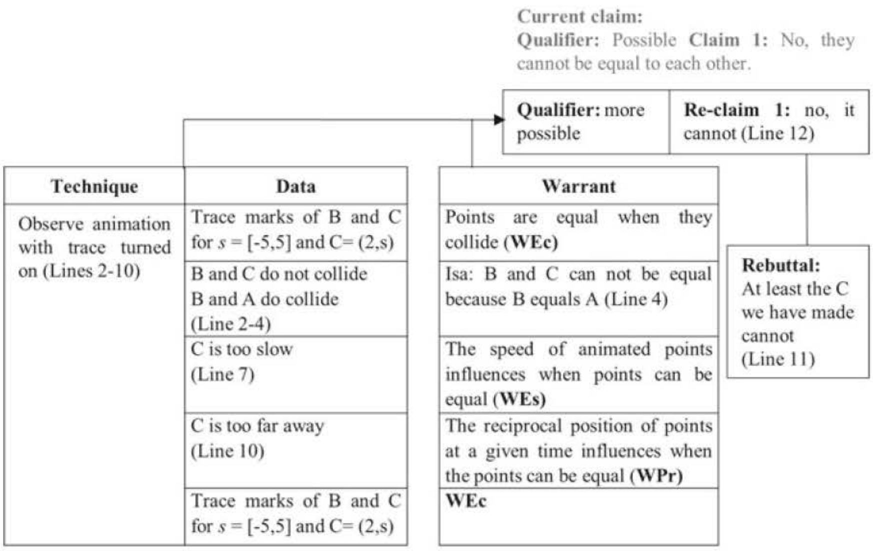

Fig. 5 Justification sub-process 1 through the lens of the analytical

Notice that the rebuttal leads to an opposing claim (line 14): "But there will probably be some that can" with the qualifier *possible*. This opposing claim becomes the students' Claim 2.

Excerpt 2: justification sub-process 2

Em and Isa gesture how the points move on the screen. Then they go back to observing the screen.

15	Isa	Okay… Ehm… Wait a minute. If we do like this. [Stops animation, edit C to (2, 2*s*), and start the animation].
16	Both	[Observe the animation].
17	Isa	Okay, so no. Why does it not? [Clicks the back arrow and C returns to C = (2, *s*)].
18	Both	[Observe the animation].
19	Isa	Hmm, let's see.
20	Em	They will never collide.
21	Isa	They will never, ehm, it can never be C equal to B because C is too slow.
22	Em	Because C moves parallel to A.
23	Isa	Yes
24	Em	And when A and B collide, then C is next to it. All the time, so it will never be able to get there, unless…
25	Isa	What?
26	Isa	No not unless. It just can't (Fig. 6).

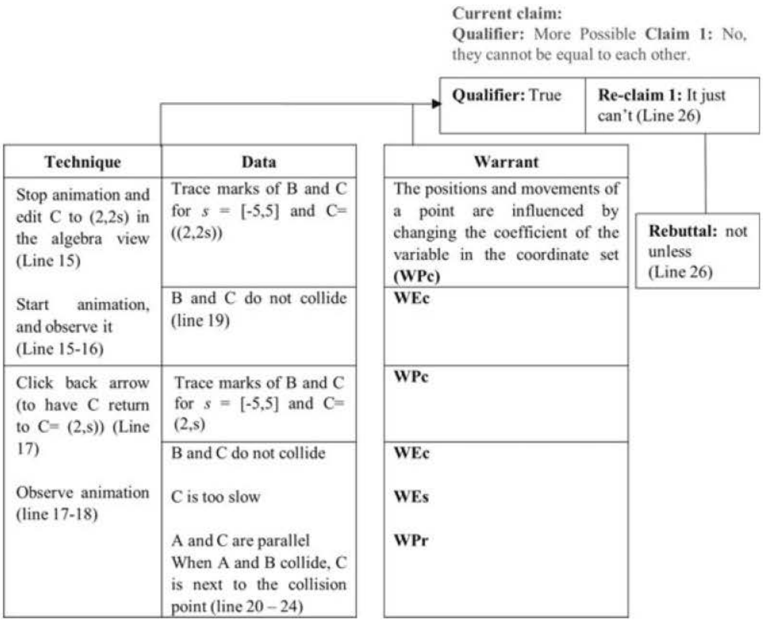

Fig. 6 Justification sub-process 2 through the lens of the analytical tool

Since claims 1 and 2 stated in excerpt 1 are opposing, this change in the qualifier for claim 1 also inflicts change in the qualifier for claim 2, which goes from *possible* to *false*.

Excerpt 3 (after an intervention of the researcher): justification sub-process 3

At this point, the first author intervenes, prompting the students to continue searching for a possible "collision", implying that she disagrees with their re-claim 1. She then guides them to stop the animation and consider the position of C. She suggests: "… try and move it [the slider] so that B collides with the trace of C", and she suggests exploring the *x*- or *y*-value for C. Then excerpt 3 follows:

27	Isa	So, let's see. Look.
28	Em	Wait a minute. Must it [point B] collide with A at the same time?
29	Isa	No not at the same time, it [point C] just needs to be parallel with A.
30	Em	Okay, okay.
31	Isa	And the lines don't need to have the same length.
32	Em	Yes.
33	Isa	[Clicks to edit the *y*-value of C—this also makes the trace disappear in the graphics view].
34	Em	Can we do like this, and then we need to move C down there [points at $(0, 2)$].
35	Isa	Yes, but how do we do that?
36	Em	Right, now it [point C] starts at two, so it starts there. Can we get it to start further down? [points at $(2, 2)$ then $(0, 2)$].
37	Isa	Oh yeah, it starts here [points at $(2, 2)$].
38	Em	Can you get it to start at minus one?
39	Isa	[Edits C from $(2, s)$ to $(-1, s)$. Starts animation.]
40	Both	[Observe the screen].
41	Em	Wait, they might collide. A still collides, so no.
42	Isa	No, but we need to change…
43	Isa	[Stops animation].
44	Em	So we get it to start a little further down, then it might do like this. [Gestures on the screen how C and B approach $(1, 1)$ to collide. C from 4th quadrant and B from 2nd quadrant]. And A then, something, it will be before.
45	Isa	Yes, but it is s we have to change.
46	Em	Is it s we have to change then?
47	Isa	Yes, can we do like this then?
48	Isa	[Edits C to $C = (2, 0.5s)$, Starts animation].
49	Both	[Observe screen].
50	Em	It [point C] is still moving parallel with A, Isa.
51	Isa	Yes, it is supposed to do that.
52	Em	B and A still collide at the same time!
53	Isa	Yes, but C is a little behind, C is half the time behind al-ways, okay. [Collision of C and B happens for $s = 2$].

54 Em So, they can do it!
55 Isa Yes!
56 Em Oh, so it was just a little too fast (Fig. 7).

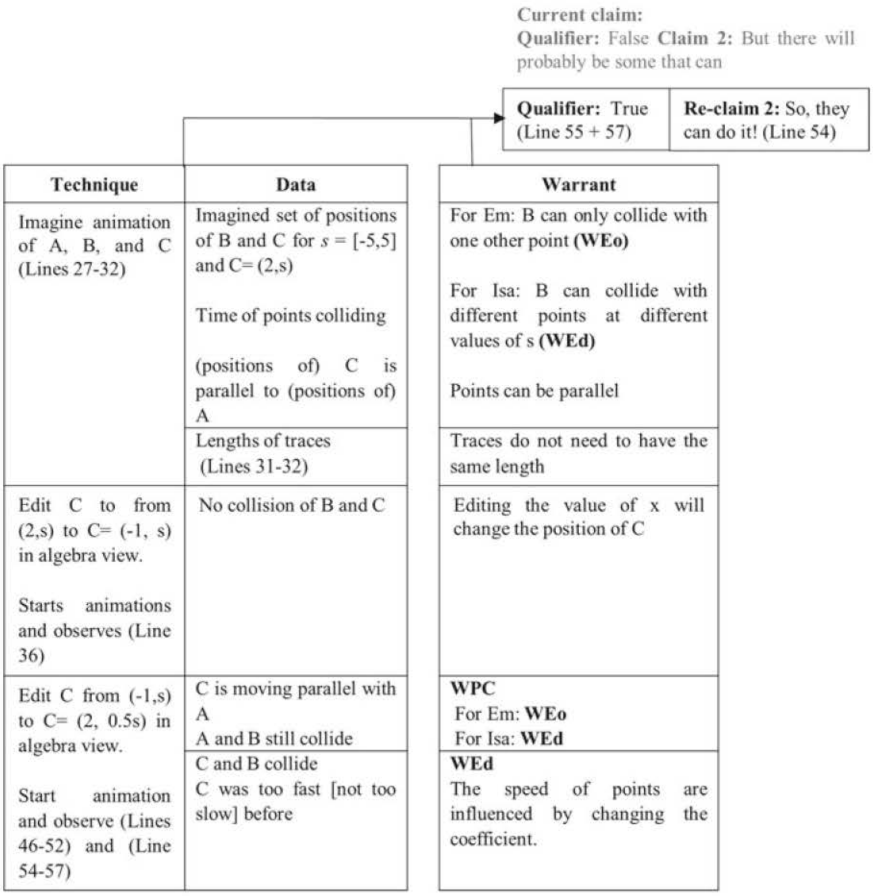

Fig. 7 Justification sub-process 3 through the lens of the analytical tool

Again, as claims 1 and 2 are opposing, the change in the qualifier for claim 2 inflicts a change in the qualifier for claim 1, shifting it from *true* to *false*.

4 Results and Discussion

In this section, we discuss findings from the analyses of Em and Isa's overall justification process. Specifically, we discuss the two main claims that the students put forth, focusing on their mutual relationship given by the changes in their qualifiers. Then, we give an overview of the students' instrumented techniques and warrants used to generate and interpret data and discuss the insight we gain on the students' reasoning competency.

4.1 Claims and Qualifier Change

Em and Isa argue for two opposing claims:

Claim 1: B and C cannot be equal.
Claim 2: B and C can be equal.

As the two claims are opposing, an increase in the qualifier of Claim 1 toward *true* leads to an implicit decrease in the qualifier of Claim 2 toward *false* and vice versa. This is illustrated in Table 1.

Table 1 The changes of qualifiers of the two claims from the initial claim throughout the three justification sub-processes

	Initial claim	Sub-process 1	Sub-process 2	Sub-process 3
Claim 1	Possible	More possible	True	False
Implicit change			↓	↑
Claim 2		Possible	False	True

Arrows indicate the implicit decrease in the qualifier of one claim in relation to the increase of the other

For Isa and Em, claims 1 and 2 correspond to two possible responses to the question in the task. Precisely what spurs Claim 2 about the possible equality of B and C is unclear. Neither student refers explicitly to the trace marks left by C and B or to their intersection. Claim 2 seems to be related to the rebuttal in line 11: "At least the C we have made cannot", suggesting that the students extend the constructed point C to all points on the trajectory $x = 2$. The students seem to be seeking phenomenological evidence of a "collision" between B and C to further convince themselves of Claim 2. As they fail to produce a collision in the second justification sub-process, the qualifier of Claim 1 shifts to true, while Claim 2 shifts to false. After the researcher's intervention, suggesting that the "collision" is possible, the students persist and eventually produce an example of C = B. The students seem to view this as evidence confirming Claim 2 and leading to the rejection of Claim 1.

4.2 Instrumented Techniques, Data, and Warrants

The students use the following two techniques (T_n):

T_1: start the animation of the slider in the algebra view, then observe the screen where the trace is turned on for all points;
T_2: edit the coordinates of a point in the algebra view, then use T_1.

T_2 can be further subdivided into:

T_{2C}: edit coefficient for a variable.
T_{2T}: edit a term in the constant coordinate.

In Excerpt 1, T_1 generates the data that leads to Claim 1 (line 3). The students describe the points as moving and the equality of points as "collision" rather than as an intersection of lines. For point C, they seem to perceive the situation differently: Isa talks about C as "too slow" (lines 7 and 21), from which we infer the warrant "the speed of animated points influences when points can be equal". On the other hand, Em describes C in relation to other points: "too far away" (line 8 and 10) "C is next to it" (line24) ("it" refers to where A and B collide). We infer the warrant here to be: "the reciprocal positions of points at a given time influence when the points can be equal". This suggests that animating the slider can give the impression of points moving along parallel and perpendicular trajectories, as discussed in the preliminary analysis of the task.

T_{2C} is used twice and T_{2T} once to produce a collision of C and B, as evidence of Claim 2. Both techniques are used in trial and error strategies. T_{2T} is first used in Excerpt 2, but it does not produce the collision. Without discussing the data generated, the students return to the original description of C. Em suggests that "because C is parallel to A" (line 22) and as "A collides with B", Claim 2 will never be possible. We infer this justification process to rely on the warrant **WEo**, "B can only collide with one other point", that feasibly emerges in Excerpt 1 from Isa's words: "No, they cannot be equal to each other. Because these [points at A then B with cursor], they can be equal to each other" (line 4). **WEo** seems to be used again twice in Excerpt 3, but not by Isa (line 29). Em seems to value this (mathematically false) warrant (lines 50 and 52); it is only when she sees the collision of B and C that she abandons **WEo**.

Conceivably, without the author's intervention, the students would have settled with Claim 1. However, such an intervention spurs the students to continue searching for evidence for Claim 2. Indeed, in Excerpt 3, Em suggests to "move it [C] further down" (lines 34 and 36), but neither student knows how to accomplish this. Em tries to use T_{2T} to do this, but as she changes the x-value, point C moves in an unexpected (for her) way, horizontally instead of vertically. By enacting T_{2C}, Em is able to edit the coefficient of s to 0.5, relying on the warrant **WPc**, "The positions and movements of a point are influenced by changing the coefficient of the variable in the coordinate set". This results in the desired collision that the students interpret as evidence strongly supporting Claim 2.

We note that the students never seem to consider the trace alone as evidence in support of Claim 2, even though the first author had given a strong hint in this direction: only one warrant seems to refer to the trace. Moreover, the students seem to refer to specific lines or trajectories (though lines are never mentioned explicitly) through the names of the points moving on them. This leads them to speak of A and C as "being parallel". However, eventually, Em refers to the *movement of A and C* as being parallel (line 50). We see this as a small step toward the distinction between "colliding points" and "intersecting lines", which we see as key in the students' potential progress in this mathematical domain.

5 Discussion

In this section, we start by discussing the specific situation of Isa and Em in relation to the reasoning competency; then, we reflect upon what is gained by the analytical tool we designed and used and on the theoretical implications of the coordination of the TIG and Toulmin' model.

5.1 How Does Isa and Em's Use of Digital Technology Influence Their Reasoning Competency?

Isa and Em engage in justification by using GeoGebra to generate, explore, and interpret data. As mentioned in the case presentation, most pairs of students did not argue for possible equality of points B and C since they only considered the point C constructed initially, without thinking about tweaking its coordinates. On the other hand, Em and Isa seem to reach a conception of C as the set of points on the trajectory $x = 2$. We see this as an essential step in Em and Isa's mathematical reasoning that allowed them to make significant advances in their exploration and reasoning.

Isa and Em's data generation is limited by the techniques they implement, primarily T_{2C} and T_1, which do not include adding a term to the expression containing the variable. Whether the data they generate is interpreted as evidence for or against a claim relies heavily on their warrants. While Em relies very much on the warrant **WEo** (B can only collide with one other point), Isa interprets the data through the warrant **WEd** (B can collide with different points at different values of s). Such warrants lead to interpretations of the data as that constitute primarily phenomenological evidence (Baccaglini-Frank, 2019) of their claims. This is also the case for warrants **WEs** (the speed of animated points influences when points can be equal) and **WPr** (the reciprocal position of points at a given time influences when the points can be equal). However, the warrant **WEd**, and more so the warrant **WPc** (the positions and movements of a point are influenced by changing the coefficient of the variable in the coordinate set), start to establish relationships between the dynamic points and the algebraic expressions.

We believe that interpreting the movements and positions of points in relation to the expressions in the coordinate sets in the algebra view could have helped activate (or construct) more profound mathematical knowledge.

5.2 Gaining Insights into Argumentation Processes in a Digital Environment

The adaption of Toulmin's model to the context of a digital interactive environment through the use of the scheme-technique duality has led to an analytical tool that sheds light on how the duality can play out in justification processes in a digital interactive environment. Our analytical tool provides the techniques as ways of generating data, inferring warrants thanks to the schemes-technique duality, and it shows how data is interpreted as evidence for a certain claim.

In particular, the tool provides structure to the observed justification processes, organizing visible elements and allowing us to make inferences about the implicit warrants and the qualifier. Indeed, through the inferred warrants, we interpret the visible parts of the argumentations and their relations that provide insight into the students' more general reasoning competency. Since a warrant is an explicit hypothesis about students' conceptions (and misconceptions) relative to the mathematical concepts they are grappling with, the students' warrants are what allow them to interpret feedback from the digital environment as evidence for their claims. For example, the warrants **WPc** (the positions and movements of a point are influenced by changing the coefficient of the variable in the coordinate set) and **WEs** (the speed of animated points influences when points can be equal) reflect the students' conceptualizations of point C, which they seem to see as a "generalized" point relative to the value of s and to the expression in the coordinate set. Such warrants allow the students to interpret the generated data as different sets of C. Through such warrants, obtaining evidence for a claim becomes a matter of generating data that "represents" the claim.

Further, the student's development and exploration of techniques empower them to generate further data in their justification processes. Techniques that involve both the graphical view and the algebraic view might further activate their reasoning competency, as we noted earlier.

5.3 Theoretical Implications of Adapting Toulmin's Model Through the Scheme-Technique Duality

We start by noting that the two theoretical approaches we consider are not symmetrical. The TIG draws on developmental psychology studied by Gerald Vergnaud and partly on cognitive ergonomics (Artigue & Trouche, 2021; Rabardel & Bourmaud, 2003). Later the construct of *technique* from the Anthropological Theory of the Didactic was adopted and reinterpreted within the TIG (Artigue & Trouche, 2021), leading to the development of the scheme-technique duality (Drijvers et al., 2013). On the other hand, Toulmin's model is not a theory but an analytical model. This makes it more flexible and applicable across sciences (Toulmin, 2003). It was originally positioned in the discipline of law (Toulmin, 2003), although its use in mathematics education has been extensive.

Despite the asymmetry outlined above, we see the conceptualization of *knowledge* as the main linkage between the scheme-technique duality and Toulmin's model. In Toulmin's model, data is observable factual knowledge that can imply more implicit knowledge in the form of warrants. In the duality, knowledge appears in terms of schemes that are partly visible, thanks to the related techniques (Drijvers et al., 2013). There is a parallel distinction between the observable and the implicit that has allowed us to link warrants to the notion of schemes and data (and its generation) techniques. Hence, we can see techniques as "windows" onto the students' knowledge about the objects at play (warrants), as they generate, notice, and interpret observable facts (data) as evidence of their claims in a justification process.

In our effort to understand the specific phenomenon of student justification processes in a digital environment, we also found ourselves in need of adapting the units of analysis. This led to the conception of sub-processes of justification within a greater process. Moreover, the sub-processes that correspond to the units of analysis capture the transition from a "claim" to a "re-claim", which is structurally different from Toulmin's original model. Indeed, rather than a static, finished argument, our sub-processes capture the formation of arguments aimed at changing the qualifier of a claim or reformulating the claim itself as a "re-claim". Such adaptation of the unit of analysis makes Toulmin's model more compliant with the scheme-technique duality.

The adaption of Toulmin's model to the context of argumentation in a digital interactive environment seems to provide a significant tool for particular types of argumentation processes that we refer to as *instrumented justification*. We conclude with the proposal of a definition for such a process.

Instrumented justification is a process through which a student modifies the qualifier of one (or more related) claim(s) using techniques in a digital environment to generate and search for data and warrants constituting evidence for such claim(s).

6 Concluding Remarks

In this chapter, we set out to contribute to a deepening of digital technology aspects of KOM's (Niss & Højgaard, 2019) reasoning competency. We have approached this by developing an analytical tool by adapting Toulmin's argumentation model through the scheme-technique duality from the Theory of Instrumental Genesis to define and capture students' processes of *instrumented justification*. The tool has provided us with a lens through which to gain insight into how the students' use of a digital environment is intertwined with their justification processes and hence with their reasoning competency. In a digital environment like GeoGebra, the students' interpretations of the objects represented are key in how the students consider them as evidence for a claim.

The theoretical developments presented in this chapter should, in our future research, be put under further scrutiny to consider how they align with other aspects of the TIG and the KOM; for example, how processes of instrumental genesis in the context of instrumented justification unfold, as well as other interesting aspects. To do this, we see potential in using a networking approach (Prediger et al., 2008) that conceives this first attempt of ours as a form of coordination between the TIG and Toulmin's model, where Radford (2008) suggests a comparison of principles, methodology, and paradigmatic questions to consider the compatibility between the coordinated elements.

Finally, from a mathematical teaching/learning point of view, the case of Em and Isa that we investigated in the chapter revealed a tension between what they referred to as "colliding points" and a yet implicit notion of intersecting trajectories. Initially, the students argued that $B = C$ was not possible as the "collision" did not occur. To overcome this interpretation, we conjecture that it is necessary for the students to reach a generalized conception of C, as *any* point on *any* vertical line instead of the specific point (e.g., $C = (2, s)$) that moves in a certain way along the vertical line. Such a generalization would entail overcoming the specific dynamic behavior of point C and conceiving its dynamism in a more general way. We see this as closely related to a broader issue of dynamism and temporality of mathematical objects, as discussed, for example, by Sinclair et al. (2009). The fine-grained analyses obtained through our analytical tool suggest that awareness of students (mostly implicit) warrants used in instrumented justification processes, and thus related to specific techniques carried out within the digital environment, can provide precious insights into their mathematical reasoning competence.

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Paper 4

Gregersen, R. M. (In review). Lower secondary students' exercise of reasoning competency: Potentials and challenges of GeoGebra’s algebra view. *International Journal of Mathematical Education in Science and Technology*

Lower secondary students' exercise of reasoning competency: Potentials and challenges of GeoGebra's algebra view

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Lower secondary students' exercise of reasoning competency: Potentials and challenges of GeoGebra's algebra view

This study addresses the potentials and challenges of algebraic features of the GeoGebra software as a means for students in lower secondary school to exercise their reasoning competency on the algebraic properties of variable points. The evolution of a task with variable points is presented in order to provide a range of students' justification processes and arguments on two different versions of a task in GeoGebra with variable points defined by variables. Student exercise of reasoning competency and their tool use in their instrumented justification process is analysed in terms of the instrumental approach and the Danish Competency framework. Results show that to draw on the algebraic tools' potential in relation to the variable points, the students must be supported in developing their tool use for both problem-solving and justification. As the feedback in GeoGebra is mostly graphic, this also extends the use of graphic features; where phenomenological impressions can be a stepping stone towards identifying core concepts. However, student experience challenges towards exercising their reasoning competency, such as restraining from trying other solving strategies when failing, referring to the failure of strategies in their justification and failing to recognize procedural mistakes expressed in the representational system.

Keywords (3–10): Reasoning competency, instrumented justification, dynamic environment, GeoGebra, algebra, variable points, lower secondary school, mathematics education

Introduction

Digital tools are an increasingly integrated part of mathematics education. In Denmark, the Dynamic Geometry System (DGS)—GeoGebra, in particular—has been implemented in primary and lower secondary mathematics education (Højsted, 2020b). Consequently, research in mathematics education in Denmark has gained attention regarding how using digital tools can influence on students' exercise and development of their mathematical competencies (Geraniou & Jankvist, 2019, 2020; Niss & Højgaard, 2019). In this paper, I continue this line of research by narrowing the focus to students' reasoning competencies (Niss & Højgaard, 2019) for lower secondary students. Indeed, how digital tools influence students' reasoning competencies and reasoning processes has been debated and researched over the past decades. Some argue that they undermine the need for theoretically validating mathematical statements (Hoyles, 2018; Laborde et al., 2006). Others take a broader perspective on reasoning processes, finding that using a dynamic geometry environment (DGE) can support explorative practices and forming conjectures in the process of reaching a theoretical proof (Dana-Picard, 2009; Mariotti, 2012; Sinclair & Robutti, 2013). Taking into consideration that the majority of students' reasoning processes in lower secondary school are usually informal ways of convincing themselves and peers that their understanding or idea for solving a problem is correct (Balacheff, 1988; Jeannotte & Kieran, 2017; Lithner, 2008), the use of a DGE can be a means to support students in exercising their reasoning competency. However, for this age group, the research on the DGE as a tool that supports reasoning processes has largely focused on geometry (e.g. Alqahtani & Powell, 2015; A. Baccaglini-Frank, 2019; A. Baccaglini-Frank et al., 2013; A. E. Baccaglini-Frank, 2012; Højsted, 2020a, 2020c; Mariotti, 2012) and, to a lesser extent, the algebraic domain, where Computer Algebra Systems (CAS) and graphing tools have been predominant. GeoGebra's unique integration of DGS features and algebraic CAS-like features (Marcus Hohenwarter & Jones, 2007) holds the possibility to give students experiences of algebraic concepts through dynamic geometric representations. The integration manifests in GeoGebra’s interface, as representations of mathematical objects, can be constructed and transformed visually in a graphic section and symbolically or numerically in an algebra section. This results in a multi-representation of mathematical relationships that can be simultaneously transformed. The algebra features in GeoGebra are accessed through a panel, the so-called a*lgebra view*, where all objects are simultaneously represented both algebraically and numerically. Representations of objects can be constructed and manipulated via an input field in this panel. This includes variables represented by a slider tool (see figure 1). The slider tool is used to differentiate the variables and "…has the potential to be an important link between geometric and algebraic representations" (Mackrell, 2011, p. 3).

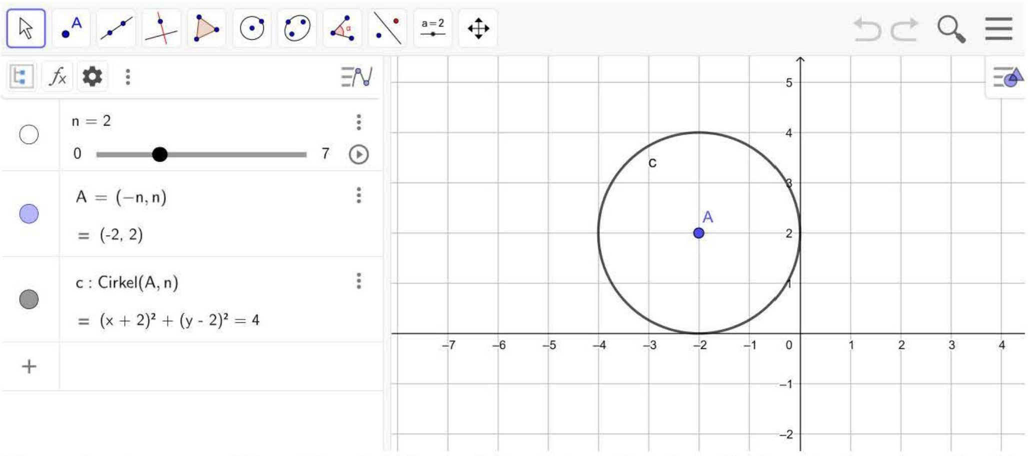

Figure 1. An open slide of the interface of GeoGebra Classic with the algebra view. In this example, the variable n is represented by a slider and defines the radius of the constructed circle c and the position of point A.

The research concerning GeoGebra's integration of DGS and algebraic features mostly addresses the learning and teaching of topic-specific areas, such as equations, functions and calculus (e.g. Anabousy et al., 2014; Aytekin & Kiymaz, 2019; Granberg & Olsson, 2015; Markus Hohenwarter et al., 2009; Stupel & Ben-Chaim, 2014). Whereas research on its potential for students to exercise their reasoning competency in relation to basic algebraic concepts (such as the generality of a variable, algebraic expressions or the equal sign) is very scarce (Gregersen, 2022). This is highly relevant as algebra in early secondary years is critical for preparing for the transition from concrete to more abstract mathematics that often serve as a gatekeeper to success in high school, postsecondary education and many career paths (Bush & Karp, 2013). In GeoGebra, points in the coordinate plane are one of the most basic geometric representations. If a point is defined by a variable and controlled by a slider, it can be considered a 'variable point'. In this paper, I hypothesize that variable points can provide students with the possibility to conjecture about algebraic properties of simple expressions on the basis of the mathematical theory of the position of points in the coordinate plane.

Therefore, this paper aims to explore which potentials and challenges can be identified for lower secondary students' exercise of reasoning competency to conjecture and justify algebraic properties of variable points (in GeoGebra) through using tools in GeoGebra's algebra view.

To explore potentials and challenges, I present the evolution of a task that has been implemented in classrooms on two occurrences, where I also draw on prior results to account for earlier identified challenges that were considered in the task's evolution. For each implementation, I analyse students' work in reasoning competency as well as their tool use of GeoGebra's algebra view as they solve and justify solutions for a task with variable points.

A Danish perspective on mathematical competence and reasoning competency

Since the beginning of the century, a competency paradigm has arisen in education policies (Geraniou & Jankvist, 2019). In Denmark, this resulted in the development and implementation of the mathematics competency framework, the KOM framework (KOM) (Niss & Højgaard, 2019). This framework aims to describe what mathematics as a discipline demands of cognitive processes in terms of competence. It sought to overcome the understanding of school mathematics as the sole learning of a subject matter to encompass a set of competencies that reflect what is distinctive for mathematics practice in society. KOM defines mathematical competence as "…someone's insightful readiness to act appropriately in response to a specific sort of mathematical challenge in given situations" (Niss & Højgaard, 2019, p. 6). The KOM framework comprises eight competencies, one of which is the reasoning competency. Reasoning competency is associated with situations in which students analyse or produce mathematical arguments. This can be either oral or written arguments and in a range of forms from exemplifying to deductive and formal proof. An argument is seen as a chain of statements linked by inference that is used to justify mathematical claims or solutions to mathematical problems (Niss & Højgaard, 2011, 2019). A competency is not disengaged from the remainder, and some competencies are more closely related as they often appear in the same context and the degree of one competency can influence the degree of another. For example, if challenged to prove a theorem, a person's symbol and formalism competency as well as their representation competency can influence their reasoning competency in the given situation. As this study examines students' uses of the algebra view, their symbol and formalism competency is particularly relevant. The symbol and formalism competency is associated with situations of using and handling mathematical symbols, symbolic expressions and transformations, as well as the rules that direct their use. The symbol and formalism competency is closely related to the reasoning competency, though distinct in the context of application:

> In principle, the ability to carry out pure routine operations, e.g. calculations, may be said to fall within the reasoning competency since it involves the justification of a calculation's result. However, what one person may regard as a routine operation, another may regard as an insurmountable problem. The actual carrying out of these operations is therefore included under…the competency dealing with mathematical symbols and formalisms, while being able to activate the operation belongs under the reasoning competency if this activation demands creativity, analysis or overview. (Niss & Højgaard, 2011, p. 61)

Likewise, the problem handling competency is particularly relevant, as problem solving constitutes the context for the student's exercise of reasoning competency. "The procedure of attaining an answer is a core element…" (Niss & Højgaard, 2011, p. 53) of the problem handling competency and is associated with situations of posing and solving mathematical problems by devising and implementing problem-solving strategies. This is closely related to the reasoning competency which concerns justifying strategies and solutions.

A person's competency is an evolving situated entity and can be developed over time through active participation in (new) mathematical situations. The degree and development of competency can be evaluated over time and also compared between individuals (Niss & Højgaard, 2019). However, this study's aim is not to evaluate progress within the students' possession of reasoning competency but rather to discover if the given context of variable points and use of tools in the algebra view can be a means for students to take active participation in reasoning processes through explicit conjecture and justifying the algebraic properties of the points. Nevertheless, as new (for the student) mathematical problems demand some development in the already possessed competency, I denote it as students exercising their reasoning competency as learners in their "…*enactment* of mathematical activities and processes" (Niss & Højgaard, 2019, p. 3).

Students' use of digital tools in reasoning processes

To analyse the student's use of the algebra view, I draw on the instrumental approach, which I unfold in the coming paragraph. Drawing on this perspective and earlier results, I deliberate on the reasoning processes that take place as students use digital tools, which I consider to be the notion of *instrumented justification* (Gregersen & Baccaglini-Frank, 2022). Finally, I point to the already-known potentials and challenges associated with using digital tools in reasoning processes.

The instrumental approach to mathematics education

The instrumental approach is a developmental theory, conceptualizing how a subject (from now on referred to as a student) learns to use a material or non-material artefact for goal-directed activities related to specific situations (e.g. using the right angle tool in a DGE to construct a dynamic square or using the slider tool to investigate the tangent slope). Such a learning process creates what is termed an *instrument* (Artigue & Trouche, 2021). The constructed instrument is distinct from the artefact. An artefact is a human product that carries human cultural and social significance, meaning that it mediates human activity (Drijvers et al., 2013). An instrument has components from the artefact and cognitive components from the student, drawing on the psychological tradition of considering tools and aids as a functional extension of the body and mind (Rabardel & Bourmaud, 2003). Undeniably, developing an instrument is not a trivial endeavour. Imagine learning to play the trumpet, drive a car or use a new piece of software. It takes time and effort to understand the mechanics and obtain fluency. This process is called *instrumental genesis.*

The process of instrumental genesis is shaped through the dual process of *instrumentation—instrumentalization.* How and what the student thinks, acts upon and knows can form, limit and direct how they may and can use the artefact. This outlines the process of *instrumentalization.* Discovering components of the artefact is part of this process (Trouche, 2004), and which components of the artefact a student can mobilize will reflect which components have been instrumentalized for a given goal and context (Trouche, 2005). In turn, the possibilities and constraints of the artefact the latitude of what is possible for the student to do and think and in which manner. This outlines the process of *instrumentation.* The dual nature of instrumentation and instrumentalization comes down to the student's thinking and conception being shaped by the artefact, but also by shaping the artefact (Hoyles & Noss, 2003; Trouche, 2004). The processes of the duality arise by mediated activity, as objects are mediated to the students through their mobilization of the instrument. The way students use components of the artefact to solve a task is termed a *instrumented technique* (Trouche, 2005). *Technique* is adopted from the anthropological theory of didactics, but within the instrumental approach, it is considered a gestural expression of the students' cognition in terms of schemes. In this paper, however, the cognitive perspective will be considered in terms of reasoning competency, and I solely focus on how students' instrumented techniques appear with their justification. The way in which instrumented techniques relate to justification processes will be discussed in the following section.

Since instrumental genesis is goal-directed, I denote the instrumentalization of instrumented techniques for the goals of problem solving and justification. A student's goal of a mobilized instrumented technique is, in reality, intertwined within the process of instrumental genesis, but will be differentiated in the analysis and discussion for the purpose of pointing to specific cases of justification. Such mediated activity can be pragmatic—aimed towards obtaining a certain outcome or change in the "world", including in the digital environment—or epistemic and aimed towards gaining insight into the objects represented in the environment (Rabardel & Bourmaud, 2003). Artigue (2010) argues that both pragmatic and epistemic mediation serve a purpose, but a predominance of pragmatic mediation has little or even negative educational value. In the context of reasoning competency and justification processes, the mediated activity's goal is to justify or verify a proposed solution, in which case, epistemic mediation serves a purpose of insightfulness that students need to obtain in order to understand the concepts in play that justify their solution.

Digital technology and justification

To coin mathematical reasoning and justification processes, I follow the very broad account laid out in KOM's definition of reasoning competency that reasoning contains the production or analysis of an argument as a chain of statements linked by inference in order to justify mathematical claims or solutions to mathematical problems (Niss & Højgaard, 2019). As earlier indicated, I focus on student justification and justification processes that relate to the solution of problems, and due to the age group I study, their justifications and final arguments will, at best, have implicit references to mathematical theory but be mostly informal in nature. It follows that the arguments the students produce do not have the structure of a formal proof (Duval, 2007), but rather that of everyday languish. To consider what constitutes an argument, I draw on Toulmin (2003's) model of argumentation, which denotes the basic elements of an argument. The elements consist of a claim along with a qualifier that indicates the likelihood of the claim (e.g. true, false, possible, unlikely), data in support of the claim, and a warrant which are inference rules that connect the data to the claim. In other words, the warrant is the logic supporting the argument. It is not necessarily but can be true in a mathematical sense. Further, the warrants in many cases are implicit and must be inferred. As Olive et al. (2010) argue, once digital technology enters an educational practice, it emerges into the practice and cannot be separated as a singular process. This also extends into student justification processes related to the solution of problems, as the digital environment becomes a catalyst in the process. To conceive such influence of a digital environment in the justifications process, Gregersen and Baccaglini-Frank (2022) attempt a definition of *instrumented justification*:

> …a process through which a student modifies the qualifier of one (or more related) claim(s) using techniques in a digital environment to generate and search for data and warrants constituting evidence for such claim(s). (p. 135)

This definition is linked to the instrumental approach through the construct of techniques as well as warrants that are likewise considered expressions of the student's schemes. The definition specifies how the use of techniques considers the basic elements in the process of forming the argument. In addition to instrumented justification, which is a process, I also consider students' final arguments. A complete or final argument is when the qualifier of a claim is considered "true" by the student and/or the student discontinues their search for data (Gregersen & Baccaglini-Frank, 2022).

Potentials of digital environments in reasoning processes in the algebraic domain

The early pioneers of technology in (early) algebraic education, such as Kaput (1992) and Confrey (1992) developed software (Simcalc and functionsprobe) that supports students' algebraic thinking as it relates to functions. Their studies revealed how dynamic, multiple representational structures that can be acted on by will can allow students to gain an understanding of algebraic concepts before having a well-founded understanding of algebraic routines and notation, as the dynamic software allows students to observe and create relationships of dynamically represented concepts (Kaput & Schorr, 2007).

A central property of dynamic environments is the continual feedback on the user's interaction with representations, which students can interpret as verifying or falsifying their ideas and attempts to solve a task (Arcavi & Hadas, 2000). Feedback by visual representations can support students in tackling existing or emerging algebraic obstacles. According to Olsson (2018), students that can analyse the feedback from GeoGebra are more likely to justify their solutions based on their mathematical knowledge. One can argue that these students can take advantage of the possibility of epistemic mediation of the artefact. However, students' reasoning can also rely on technology without questioning the applicability and reasonableness of the feedback or results, which considerably limits the epistemic mediation (Nabb, 2010).

Specifically for GeoGebra's algebra view, a limited amount of studies point to the slider tool for providing feedback that dynamically links graphic, numeric and algebraic representations of a variable. For example, dragging a slider represents the variation of a numeric value, which allows students to verify conjectures about the mathematical relationships of a variable. However, if the slider tool is only represented visually to dynamically transform graphic and numeric representations, the symbolic representation of relationships is only expressed through the structures of the graphic representation. Therefore, in supporting students' conjecturing explicitly as pertains to relationships of the variable and expression, they can be given access to create and transform the symbolic representation (Gregersen, 2022). This potential has historically been explored in the software, Logo (Laborde et al., 2006).

Method and data collection

There are two main variables at play when exploring the potentials and challenges for students' use of tools in algebra view in GeoGebra in order to exercise their reasoning competency concerning algebraic relationships: the student's use of the digital tool and their cognition in terms of competency. To obtain empirical data, a sequence of tasks has been developed and implemented in classrooms on two occasions. From the set of tasks, I report the evolution of the Equal Points task as an analysis of the first implementation. This indicated that the Equal Points task had the potential to provide a context for students to reason regarding the algebraic properties of variable points.

However, the task was inaccessible for most students, and to get further insight into potentials and challenges, it needed to be developed. First, I report on the task's settings and present the Equal Point task. Analysis 1 the potentials and constraints of GeoGebra and variable points in terms of the instrumental approach is analysed. Then to unfold the student challenges I conduct a global analysis of students' exercise of reasoning competency, reports of earlier results along with a case of a pair of students' instrumented justification. The analysis is synthesized to argue for the evolution of the Equal Points task.

In the second analysis, I focus solely on students' work on the second edition of the Equal Points task. Again, I present a global analysis of grouped student answers to gain an overview of the outcome of the experiments in terms of students' use of reasoning competency and differences in the nature of their justifications. Based on an information criterion (Flyvbjerg, 2006), cases discriminate on the nature of the justification and illustrate different challenges and potentials. The cases are analysed in terms of instrumental genesis and techniques, to gain insight into students' challenges in their instrumentation of tools in the algebra view for justifying.

The two occasions of classroom experiments were conducted in lower secondary classrooms. In the first session of classroom experiments, a sequence of tasks was assigned to students in pairs in three 7^{th} grade classrooms. The experiments were done in one class at a time, in two 90-minute sessions. Seventeen pairs of students were recorded as they worked on the tasks, capturing their screen, upper body and face, and voices. The experiments were conducted in close collaboration between the author and the mathematics teachers in the three classes. All students had prior experience using GeoGebra.

In the second session of class experiments, as a pilot study, one pair of 7^{th} grade students in an interview setting was recorded in a similar fashion as the first experiment. Then, the set of tasks was assigned in a 7^{th} grade classroom where 10 pairs were recorded in one 90-minute session that the author conducted. These students had little experience using GeoGebra. Finally, one class from the first round of experiments, now in 8^{th} grade, was also assigned the tasks and recorded. The class mathematics teacher conducted this experiment after I provided online guidance. Collectively, 18 pairs of students were recorded.

Tasks and student transcripts have been translated into English from Danish.

Task evolution and emerging potentials

The development of the first sequence of tasks relied on prior research in the field. The evolution of the tasks relied, to a larger degree, on the analysis of student work on the tasks. In the coming analysis, the development and evolution will be elaborated on and centred around one specific set of tasks: the Equal Points task.

Setting the stage of the task

The overarching aim of the set of tasks was to support students in exercising their reasoning competency through the hypothesized potentials of the algebra view (Gregersen, 2022). Consequently, the task sequence had to provide a context for the student's instrumented justification and the mathematical concepts at play had to support the overarching aim by taking into account the challenges that students face in mathematical reasoning.

Mathematical reasoning is not easy to learn and master. Many studies have shown that students can struggle with both identifying the relevant properties and structuring a mathematical argument (Duval, 2007; Harel & Sowder, 2007). Hence a prerequisite is that the students have acquired the mathematical knowledge needed to solve a problem and justify the solution. Still, students do not necessarily rely on their mathematical knowledge in their problem solving and justification of solutions. Instead, they might rely on authorities to argue for truth, such as textbook examples and guides, standard formulas, teacher statements and guidance or technology (Harel & Sowder, 2007; Lithner, 2008; Misfeldt & Jankvist, 2018; Nabb, 2010). To counter these challenges, the mathematical concepts should be within the realm of the students' routine work and knowledge, but in mathematical problems that are not standard or considered routine. In addition, the mathematical concepts in the problems should be well known to a lower secondary school student, and the concepts should be possible to represent in both the algebra view and graphic view in GeoGebra and to be manipulated in the algebra view. Therefore, for the choice of the mathematical content knowledge and the types of tasks, I both considered the students' knowledge and the accessibility of the concepts in the algebra view.

In Denmark, Cartesian Geometry is introduced and used from early grades onwards (Ministry of Children and Education 2019). It is also incorporated in the logic structures of GeoGebra so that constructing points in the coordinate plane is accessible to students and is possible for them to represent and transform algebraically. Plotting coordinates into a coordinate system is, for the most part, a trivial task for a 7^{th} grade student. From the 7^{th} grade onwards, students are introduced to linear and non-linear equations as well as graphical representations in the coordinate system. Linear equations and functions are, however, new concepts for the students, and cannot be considered well-known mathematical knowledge. It follows that the task should be developed with respect to these two poles, points in the coordinate plane and the linear equations and functions. To satisfy the need for non-trivial tasks, the main mathematical object is variable points in the coordinate plane, which are points defined by expressions containing variables, allowing students to draw on their knowledge of points in the coordinate system. The variable points are hypothesized to provide a context for the students to be able to conjecture and justify the algebraic properties represented by the variable points. In the task to be presented, the algebraic properties concern equality, variables and expressions in the coordinate sets.

THE INITIAL EQUAL POINTS TASK

From the first sequence of tasks, the analysis of student answers to one specific task, the Equal Points tasks, led to considerable evolutions in the whole sequence of tasks. An example of how students work on the Equal Points task would appear in GeoGebra is depicted in figure 2. Note that points $A = (1,s)$ and $B = (s,1)$ are given in an earlier task, and point C is constructed in problem b), which has several solutions; however, $C = (2,s)$ is the most common among the students. The problems posed along with solutions in brackets in the task are:

(a) When does $A = B$? (*When* $s=1$). Justify your answer.

(b) Construct a new point C dependent on s that moves in parallel with A (*e.g.* $C=(2,s)$).

(c) Can $C = A$? If so, when? (*No*). Justify your answer.

(d) Can $C = B$? If so, when? (*Yes, several solutions. In general form* $C= B$, *if* $C= (d,\frac{1}{d}s)$ *or* $C = (d, s - (d-1))$, *when* $s = d$). Justify your answer.

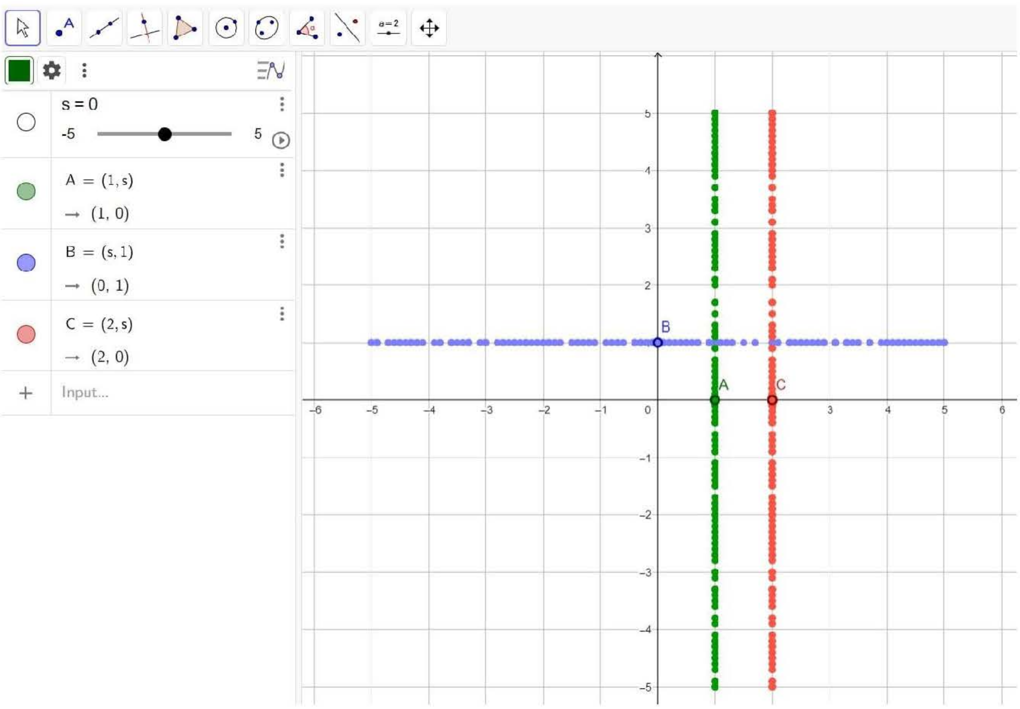

Figure 2. An example of how the Equal Points task appears in GeoGebra Classic.

Analysis 1: Instrumentation and initial potentials and challenges of the Equal Points task

First, I analyse the artefact constraints and possibilities of variable points and the associated tools in GeoGebra's algebra view. Then follows a global analysis of the student's reasoning competency in the first cycle of experiments by grouping the student's answers according to similarities in their answers and coding them according to the nature of the justification. I then summarize some earlier results that had an impact on developing the task and supply these results with an analysis of instrumental genesis.

The constraints and possibilities of variable points and the associated tools in the algebra view

By considering a given artefact's possibilities and constraints, we can get an understanding the instrumentation proces. Recall from the introduction that the algebra view in GeoGebra is accessed through a panel. Via an input field in this panel, representations of objects can be constructed and manipulated, including geometrical and algebraic representations. A component of the algebra view is the slider tool (figures 1 and 2) that vary explicitly appearing variables (as opposed to implicit appearing variables that are transformed by dragging objects across the graphic view (Gregersen, 2022)). In the Equal Point task (figure 2), students construct and manipulate variable points as defined by a variable and simple expressions. As the variable is changed, the variable point "moves" on trajectories of line segments on the plane. Hence, the coordinates of a variable point do not just refer to a single point on the plane but to a set of points restricted by the limits of the variable (in GeoGebra, the limit [(-5),5] is set by default). Sets of points can be represented by activating the *trace function* of the variable point, leaving a trace of points in the coordinate plane when the slider is dragged. If the slider is animated (i.e. it changes the numeric value automatically), the dependent point appears to move at a certain speed. The movement of points and traces appear as line segments, and therefore parallelism and intersection are indirectly represented by the movement or the trace of two points. If points intersect, they can be perceived as equal points when they occupy the same coordinate 'at the same time' as the points are defined by the same variable and time relates to the variable's value. The variable in the coordinate set can be part of an algebraic expression defining the position and length of the sets of points. Changing a constant will change the position of the point and the trajectory on which it moves. Changing a coefficient will also change the trajectory's length and if the slider is automated, it can be perceived as a change in the speed by which the point moves.

Global analysis of students' exercise of reasoning competency

In table 1, the students' answers in the video material for tasks a, c and d were condensed into a simple form and then grouped ad hoc according to similarity. The final arguments have been coded according to the nature of the justification referring to either A) an algebraic relationship, P) phenomenological impressions, N) numeric information or G) geometric properties. The students selected "No answer" if they did not get to the task for any reason. Students selected "Irrelevant answer" if they had technical issues, misunderstood the task or overlooked essential information in the task, and were not able to answer the actual problem as a result.

Table 1. Grouped student answers of a, c and d (as b is a construction task it is not included) for t he initial Equal Points task and the nature of their justification, referring to either A) an algebraic relationship, P) phenomenological impressions, N) numeric information or G) geometric properties.

Grouped student answers, n = 17 pairs	n pair(s)	Nature (A, N, P, G)
a) When does $A = B$? (*When $s = 1$*) Justify your answer		
Yes, when $s = 1$, then both points have the same coordinate set	9	A
Yes, because both points are at (1,1) when they intersect	4	N/P
Irrelevant answer	1	-
No answer	3	
c) Can $C = A$? If so, when? *(No)* Justify your answer		
No, as the points move parallel	8	G
No, as they have different x-coordinates	1	A

No, they will never have the same coordinates	1	P
Irrelevant answer	2	-
No answer	5	-
d) Can $C = B$, and if so, when? (*Yes, several solutions. In general, form C= B, if C=* $(d, \frac{1}{d} s)$ *or C =* $(d, s - (d-1))$*, when* $s = d$) Justify your answer		
Yes, when $s = 2$ and $C = (2,0.5s)$, because then C is slower	1	A
No, because $B = A$, then $B \neq C$	2	P
No, because they are never at the same point	2	P
No, because they never have the same coordinates	3	P
Yes, because their traces intersect	2	P
Irrelevant answer	2	-
No answer	5	-

The students who can establish a relationship between some phenomenological impressions from the graphic view to the algebraic information in the algebra view in their justification are exhibiting a more developed reasoning competency than students who only refer to numeric observation or phenomenological impressions. Student justifications referring to geometric relationships are incomparable in this context and can be considered a flaw in the task that deflects students' focus from the task's overall aim.

In task a), 13 pairs of students were able to recognize the equality of points A and B as having the same position in (1,1), and nine pairs were able to relate this to the variable s. These tasks seem to hold some potential. In task c), the students were able to recognize that A and C could not be equal but are mostly justified on the geometric property of parallelism in the movement of the points. This problem does not support the students' to justify with algebraic relationships. In task b), all students constructed a point C that was not equal to point B for any value of s. Following in problem d), all but one pair of students answered that $C = B$ was not possible. This problem seems inaccessible for the students due to the overwhelming number of wrong answers. However, the problem-solving process of the one pair of students who did solve the problem was further scrutinized by Gregersen and Baccaglini-Frank (2023). The results are summarized in the following section, and a complementary analysis of their instrumental genesis is presented here.

Results from earlier research and a complementary analysis of instrumental genesis

The pair of students, Isa and Em, recognized that if they changed point C while maintaining its parallelism to A, it would be possible for C to equal B.

Em and Isa's process of instrumented justification was analysed by coordinating the scheme-technique duality (Drijvers et al., 2013) with Toulmin's (2003) argumentation model, upon which Gregersen and Baccaglini-Frank (2022) elaborated. This analysis revealed that understanding C as a set of points was essential for the students' exploration and a prerequisite for the pair to start editing point C in the algebra view. However, Isa and Em's justifications mostly relied on the phenomenological experiences of the points moving in the graphics view, which appeared when activating the animation of the slider for the variable s, and to some extent the traces of the variable points. The animation and trace allowed the students to identify relevant data, such as new positions, and a change in the speed of point C in supporting their evolving claims. However, the students also missed significant possibilities in interpreting the trace function, which will be elaborated on later. Finally, it should be noted that an intervention from me, acting as a teacher, spurred the pair to continue searching for a solution (see appendix).

The following extracts of the same pairs of students' instrumental genesis are analysed with a focus on the specific moments of instrumentalization as they edit the algebraic expression in the coordinates. Then follows a description of how these insights, together with findings from Gregersen and Baccaglini-Frank (2022) have led to task developments.

Isa and Em are faced with the problem of making two points with intersecting trajectories be equal for the same value of a variable s. Further, they are expected to justify their solution, which presupposes that they understand their solution. Consequently, both pragmatic end epistemic mediation is needed in their tool use. This is a new kind of task for them, which means they must enter into a process of instrumental genesis where they apply their knowledge of the tool and the mathematical concepts at play, which is instrumentalization. Isa and Em have three instances of instrumentalization where they apply techniques in the algebra view (a full transcript of Isa and Ems's solving process of problem d can be found in the appendix). First, they edit C by changing the coefficient of s such that $C = (2, 2s)$, and start the animation.

Isa	Okay.. ehm… Wait a minute. If we do like this. [Stops animation, edit C to $(2,2s)$, and start the animation]
Both	[Observes the animation]
Isa	Okay, so no. Why does it not? [Clicks the back arrow and C returns to $C= (2,s)$]

In this instance, the pair only has a pragmatic mediation. They change the coefficient to 2, which does not make $A = B$ for any value of s and, without determining how the change in the coefficient influences point C, they return to the previous definition. Had they done so, it could have revealed the phonological impression that point C now 'moves' faster on the trajectory and that the length of the trajectory has doubled. After this instance, the girls are inclined to give up, but an intervention from me keeps them searching for a solution. Following this, they edit the constant in the y-coordinate from $(2,s)$ to $(-1,s)$ and start the animation:

Em	Can we do like this, and then we need to move C down there.
Isa	Yes, but how do we do that?
Em	Right, now it [point C] starts at two, so it starts there. Can we get it to start further down?
Isa	Oh yeah, it starts here.
Em	Can you get it to start at minus one?
Isa	[Edits C from $(2,s)$ to $(-1,s)$. Starts animation.]
Both	[Observes the screen]
Em	Wait, they might collide. A still collides, so no.

Em wants to change the position of point C. By "move the point down", it is unclear if she means down the x-axis or down the y-axis, but Isa changes the x-value so that the point moves down the x-axis. They recognize that this changes the trajectory of C to be on $x = -1$, but still $C \neq B$. Again, they do not seek to understand why this does not supply a solution, and hence still only pragmatic mediation occurs. Had they sought to move the point down to -1 on the y-axis, they had reached a solution. This would have required using the technique of adding a term to the expression, a component of the artefact that the girls possibly have not instrumentalized yet contrary to changing the already existing constant of the x-value. Finally, Isa edits $C = (2,0.5s)$, then starts the animation and finds that the points collide:

Isa	[Edits C to $C = (2,0.5s)$. Starts animation]
Both	[Observes screen]
Em	It [point C] is still moving parallel with A, Isa.
Isa	Yes, it is supposed to do that.
Em	B and A still collide at the same time!
Isa	Yes, but C is a little behind. C is half the time behind always, okay, okay. [Collision of C and B happens for $s = 2$]
Em	So, they can do it!
Isa	Yes!
Em	Oh, so it was just a little too fast.

In this instance, the pair has both pragmatic end epistemic mediations as they can relate the coefficient of s to the speed of point C. Recall that earlier analysis identified that the girls overlooked a potential insight from the traces. This becomes clearer in this transcript. As the possible position for equality between the points is indicated by the intersection of the traces of points C and B, Isa and Em could have identified the coordinates (2,1) as the position for $B = C$. This could have given the insight that C's y-coordinate should be 1 when $s = 2$, and possibly lead to a deeper understanding of why changing the coefficient to 0.5 is a solution.

The techniques the pair instrumentalize can be categorized as either T_C: edit coefficient of a variable or T_T: edit a term. Using the techniques seems to be a somewhat random trial-and-error approach. However, the slider's animation functionality is consistently used to verify or reject their trials. Yet the girls' engagement in editing the expressions in the coordinates can be considered a vital step towards exercising their reasoning competency concerning algebraic relationships.

Task evolutions addressing challenges from analysis 1

The following issue is a synthesis of the global reasoning competency analysis, the prior results on Isa and Em's instrumented justification and the analysis of Em and Isa's instrumentalization, which acted as the basis for the evolution of the Equal Points task:

A. The prerequisite of considering *C* as a set of points that can be changed made the task inaccessible for most students.
B. Using the animation functionality of a slider can supply a phonological impression of the speed of points as being dependent on the coefficient, supporting students to discover this relationship. However, we cannot expect the student to instrumentalize the animation functionality without guidance to do so.
C. Using the trace functionality on points has the possibility of providing phonological impressions of changing the coefficient in terms of the length of the trajectory and on the intersections of traces to indicate the coordinate position when points are equal. However, students need to instrumentalize the trace functionality.
D. As the students are not used to editing expressions containing variables in coordinates, their instrumental genesis for this type of task should be further supported.
E. Tasks with parallel moving points can deflect students' justification of geometric properties.

Evolution of the Equal Points task

The second edition of the Equal Points task is presented in figure 3. The following adjustments have been made to accommodate the issues.

To tackle issue A, the construction of point *C* was replaced with a predefined point: $A2 = (1, s - 1)$, omitting the need for students to recognize the possibility of changing point *C*. *A2* moves at the same trajectory as point *A* (now *A1*). Consequently, the students are only dealing with one intersection point for problems a, b and d. This also eliminates issue E. Further, it creates a situation for students to change the *y*-value for a solution to $A2 = B$ and gain experience with expressions that contain a term. This supports issue D. Also, to tackle issue D, the problems were reformulated to be more specific towards editing the coordinates. In introductory tasks, students are introduced to both the trace and animation functionality (issues B and *C*).

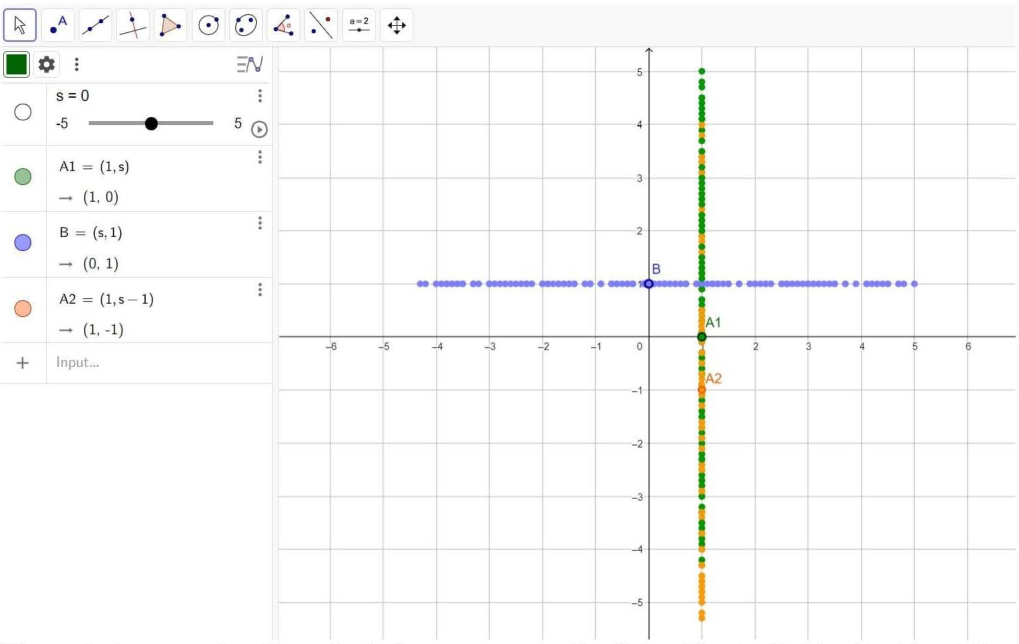

Figure 3. An example of how the task can appear in GeoGebra Classic. Points *A1*, *A2* and *B* are given in a prior task.

The problems posed (*along with solutions*) in the task are:

(a) Can *A1* = *B* and when? *(Yes, when s = 1)*. Justify your answer.

(b) Can *A2* = *B* and when? *(No)*. Justify your answer.

(c) If you change the *x*-coordinate of *A2*, is it then possible for *A2* = *B*?

- If yes, when and why? *(Yes, there are more solutions e.g. if A2 = (2,s−1) and s = 2).*
- If no, why not?

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(d) If you instead change the y-coordinate of *A2*, is it then possible for *A2* = *B* without *A2* = *A1*?

- If yes, when and why? *(Yes, A = (1,2s−1) or A =(1, −s −1)).*
- If no, why not?

Analysis 2: Potentials and challenges of the algebra view for students' exercise of reasoning competency revealed by the evolution of the Equal Points task

As in Analysis 1, a global analysis of grouped students' answers and the nature of their justification is completed to gain insight into their exercise of reasoning competency. They are presented in table 2. Again, the coding of the nature of the justifications is A) an algebraic relationship, P) phenomenological impressions, N) numeric information and G) geometric properties. Students who never got to the task for various reasons are represented by "No answer". Students who had technical issues, misunderstood the task or overlooked essential information in the task and were therefore unable to answer the actual problem are represented by "Irrelevant answer".

Table 2. Grouped student answers for the second edition of the Equal Points task and the nature of their justification referring to either A) an algebraic relationship, P) phenomenological impressions, N) numeric information or G) geometric properties.

Grouped student answers, n = 18 pairs	n pair(s)	Type (A,N, P,G)
a) Can *A1* = *B* and when? *(Yes, when s = 1).* Justify your answer.		
Yes, when s = 1, as *B* = (s,1) and *A1* = (1,s), and when s is one, they both are (1,1)	2	A
Yes, when s =1 because then both points are (1,1)	1	A
Yes, when both coordinate sets are (1,1), no justification	3	-
Yes, as the points cross each other	1	P
Yes, when s=1, no justification	3	-
Irrelevant answer	7	-
No answer	1	-
b) Can *A2* = *B* and when? *(No).* Justify your answer.		
No, as *A2* is always one below *A1* because of "the -1" (and *B* = *A1*)	2	A
No, as there will always be one point that has a distance to the intersection when the other one is at the intersection	3	P

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No, they are never at the same place at the same time	1	P
No, no justification	4	-
Irrelevant answer	7	-
No answer	1	-
c) If you change the *x*-coordinate of *A2*, is it then possible for *A2 = B*? • If yes, when and why? *(Yes, when (x,s-(x-1)), e.g. A2 = (2, s−1) and s = 2)* • If no, why not?		
Yes, when *A2* = (2,*s*-1), because then both points can have the *x*-value of 2 when s is 2	2	A
Yes, when *A2* = (2,*s*-1), because then *A2* and *B* have the same distance to the intersection of the trajectories	1	P
Yes, when *A2* = (2,*s*-1), because *A2* and *A1* are then on different trajectories	1	P
Yes, when *A2* = (2,*s*-1), no justification	6	-
Irrelevant answer	7	-
No answer	1	-
d) If you instead change the *y*-coordinate of *A2*, is it then possible for *A2* = *B* without *A2* = *A1*? • If yes, when and why? *(Yes, endless solutions when y=1, e.g. A2 = (1,2s−1))* • If no, why not?		
Yes, when *A2* = (1,2*s*-1), no justification	1	-
No, because *B* always has a distance to the intersection of *B* and *A2*'s trajectories when *A2* is at the intersection	2	P
No, because they are equal in (1,1) where *A1* = *B*, and we cannot find a solution where *A2* does not equal *A1*	4	A
Yes, when *A2* = (1,*s*1) or *A2* = (1,*s*), no justification	2	-
No, no justification	1	-
Irrelevant answer	7	-
No answer	1	-

For problems a, b and c, the students who provided an answer to the problems were able to give the correct solution. The justification of the solutions is more diverse than in the first session. A few pairs of students (a = 2, b = 2, c = 1) drew on the information in the algebra view and their knowledge of variables or coordinates in their justification. Some pairs, as with Em and Isa, drew on phenomenological impressions of the graphic view to justify their solution. In problem c, all students who attempted to solve the task found the correct solution on their first or second try. It seems that determining the value of the *x*-coordinate to obtain equality between the points was easy for the students. Two pairs of students were able to justify their solution by referring to the value of the x-coordinates, and a couple of the justifications were phenomenological in nature. Most simply did not justify their solution.

On the contrary, problem d was a struggle for all pairs of students. Two pairs did not recognize that, in their solution, $A1 = A2$, which might be related to a lack of knowledge about expressions, e.g. knowing that $1s = s$. However, four pairs who were not able to find a solution could justify why their solution did not hold due $A1 = A2$ in their solution attempts. Two pairs provided phenomenological justifications for their failed attempts. Only one pair of students, Max and Sam, found a correct solution, and it should be noted that this was obtained after I intervened. This is elaborated upon further below. What is notable is that the more direct formulation of the problem directed more students to use the algebra view to edit the expression in the y-coordinate, thereby revealing a diversity in the techniques in students' attempts to find a solution. Gaining insight into these (mostly fruitless) techniques and how the students instrumentalized them can serve to gain further insight into the potentials and challenges of the algebra view for the students' instrumented justification with algebraic properties.

Global analysis and cases of diversity in using techniques

The techniques identified can be classified as follows, and the distribution of their use can be seen in table 3:

- T_T: Edits the term only.
- $T_{C\text{-}T}$: Deletes term, then edits coefficient.
- T_{C+T}: Edits coefficient without deleting the term.

Table 3. An overview of students' uses of techniques in the processes of solving problem d, in relation to grouped answers.

Technique	In relation to grouped answers (n)	n
T_{C-T} then T_{C+T}	Yes, when *A2* = (1, 2*s*-1), no justification (n = 1)	1
T_T	No, because *B* always has a distance to the intersection of *B* and *A2*'s trajectories when *A2* is at the intersection (n = 2) No, because we could not find a solution where *A2* does not equal *A1* (n = 3)	5
T_{C-T}	No, because we could not find a solution where *A2* does not equal *A1* (n = 1) Yes, when *A2* = (1,*s*1) or *A2* = (1,s), no justification (n = 2)	3

T_T is the most used technique. Five of the nine pairs of students that answered problem d used T_T and did not consider any other options. Furthermore, these students seemed to be explicitly opposed to adding a coefficient to the expression (note the bold marking in the following transcript). This is exemplified by Bob and Dan, who from the start reject the possibility:

Bob So. The *y*-coordinate. We will have to change the subtraction, **but not *s*.**
[edit *A2* = (1, *s*-0), drags slider for *s*].
Can you change it to 0? [observes screen] Ahh, okay - We need *A2* to be equal to *B*, but these two cannot [points at *A2* and *A1*, drag the slider for *s*].

Dan Mhh, yes

Bob But - I would like to set it to zero, but then the problem is that these two [points at *A2* and *A1*] are the same.

Dan Can't we just move it back?

Bob I guess we can change it to addition.
I just set it to minus 2 [edits *A2*= (1,*s*-2), drags slider for s].
Plus 2 [edits *A2*= (1,*s*+2) drags slider for s].
I do not think it is possible. They have to meet in (1,1), but if it's "add two", it is two ahead. Unless you say "add zero".

Dan Are you allowed?

Bob No, then *A2* equals *A1*.

Bob [edits *A2*= (1,s-1) drags slider for *s*] I do not think it is possible.

After a trial-and-error approach to changing the term and using the dragging functionality for verification, the pair give up, as this only produced a solution where *A2* is equal to *A1*. Bob, in particular, has both pragmatic and epistemic mediations in his tool use as he analyses the situation according to the expression and position of the point and an expected intersection in (1,2). This leaves him able to justify why changing the term is not a productive technique, but are not able to consider other strategies or techniques. Despite the students having been introduced to changing the coefficient, Bob and Dan have not instrumentalized this technique and only consider technique for changing the term.

Four pairs of students use the $T_{C\text{-}T}$ technique. These pairs can be divided into two groups in their use of this technique. Two pairs delete the term, edit the coefficient to 1 and take it as a solution without justifying their solution. They do not recognize that *A2* = *A1*, neither in the algebraic form nor in the graphical representation where the points will move continually on top of each other. The other two pairs of students use the $T_{C\text{-}T}$ by repeatedly changing the coefficient approaching 1, e.g. Max and Sam change the coefficient repeatedly in the following order: 3, 2, 1.5, 1.3, 1.2, 1.1. For the pair to continue, I intervened for them to adjust their technique. When I approached them, I explained why approaching 1 does not solve the problem as they end up with *A2* = *A1*. I suggested that they go back to the initial definitions of *A2* = (1, *s*-1) and continue trying to find a solution without deleting *-1*. In their next attempt, they use T_{C+T} typing *A2* = (1, 2*s*-1) to reach a solution.

Discussion of potentials and challenges of the algebra view for students' exercise of reasoning competency

The components of the artefact the students instrumentalize for epistemic mediation impact the nature of their justification. Therefore, to uncover the potential of the algebra view to exercise the students' reasoning competency within the algebraic domain, I scrutinize the challenges of the students who do not consider the algebraic relationship at play in the task in their justification, but also contemplate the potentials still evident and how the potentials can be further supported in the task.

Justifying the grounds of phenomenological impression - challenge or potential?

In both the initial and second Equal Points tasks, a substantial amount of students justified their answers on the phenomenological impressions of the variable points in the graphic view, despite using techniques in the algebra view. We can consider this through the process of instrumentation as the constraints and potentials of the artefact situate what the students can do and think. The feedback the students get from using techniques in the algebra view is the change in the positions and movements of the points in the graphic view as they drag or animate the slider for a variable. For the students to make inferences about the changes and hence the relationship to the variable points a technique reveals, they must also first make sense of the representation of the objects in the graphic view. This, of course, involves epistemic mediation in students' instrumentation of the graphic view. As the data shows, very few students make the connection between the phenomenon they notice in the graphic view to the coordinate set and the algebraic expressions in the algebra view. The student who does can be considered to have a more developed reasoning competency for the given situation. However, some of the phenomenological justifications do imply an understanding of the relationship between the coordinate set for the variable points and equality. For example, for problem c, the answer is 'yes, $A2$ = (2,s-1), because then $A2$ and B have the same distance to the intersection of the trajectories'. This statement implies that the students understand there is a relationship between changing the x-coordinate and the position of the trajectory of the point left by the trace, and moreover, that the intersection of these trajectories indicates possible equality between the points. The next step is for the students to instrumentalize components of the algebra view for justification. Or in other words, they must evolve their use of the algebra view to also encompass epistemic mediation for the goal of justifying their answer. In that sense, some of the justifications that rely on phenomenological experiences can be a stepping stone for students exercising their reasoning competency in the algebraic domain and a potential. However, not all of the justifications relying on phenomenological impressions can be considered as such. For example this (faulty) justification for problem c: "Yes, when $A2 = (2, s\text{-}1)$ because $A2$ and $A1$ are then on different trajectories". This justification indicates that students understand that the constant in the coordinate set corresponds to the position of the trajectory of the point, but they fail to relate it to equality, in general, and between the relevant points B and $A2$. So, despite the fact that the students do refer to the coordinate set, these students struggle with identifying the core concepts of the problem and exercise a less developed reasoning competency. As they are struggling in their instrumentation of the graphic view, the introduction of the algebra view adds yet another layer of mathematical complexity, and the phenomenological justification itself is not a stepping stone, but rather a challenge.

Challenges related to symbol and formalism competency and instrumentalization

Students struggling with instrumentalizing relevant techniques in the algebra view can, in some instances, be related to their symbol and formalism competency, which might capture the issue for many of the pairs struggling with problem d.

We see this in the case of Bob and Dan who, in task d, are opposed to even considering "changing" s, which could be multiplying by a coefficient or by -1. The students' handling of symbols becomes a challenge that results in a faulty answer despite the pair and Bob, in particular, having instrumentalized the technique for both problem solving and justifying. This particular difficulty could be met in a third edition of the task by adding a (visible) coefficient in the definition of $A2$.

Another example of how students' lack of symbol and formalism competency challenges their exercise of reasoning competency is those students who do not recognize that $A2 = A1$ in their solution. Their lack of conceptual knowledge going into the instrumentation process indicates that, as they type in $1s$ in the coordinate set of $A2$, they do not recognize that s and $1s$ are the same expressions. As previously mentioned, these students do not justify their solution, which indicates that there has been no epistemic mediation in their use of the graphic view. Had they done so, they might have discovered that the two points occupied the same position on the same trajectory in the graphic view, indicating that $A2 = A1$. For these students, the potential of discovering the equality between the two expressions is then also lost. Hence, this is also an example of how a lack of instrumentation of the graphic view challenges students' instrumentalization of tools in the algebra view and ultimately in their exercise of reasoning competency.

Challenges related to the problem handling competency

Recall from the analysis of students' techniques applied for solving problem d that the majority only make use of one technique, and if that fails, they conclude that $A2$ and B cannot be equal. This can be associated with the lack of symbol and formalism competency, as just discussed. Yet, this can also be associated with a lack of the problem-handling competency. Part of this competency is to devise and implement strategies for solving mathematical problems. The strategy these students implement is to pick a technique and stick to it. Again, in the case of Bob and Dan, Bob can explain why their technique does not lead to equality, but they still stick to the chosen technique. This reluctance to explore other techniques is a challenge for students' processing of instrumentalizing other techniques in the algebra view for both problem solving and justification. Whether the cause is a lack of students' symbol and formalism competency or problem-handling competency, it poses a fundamental issue for students' exercise of reasoning competency when using GeoGebra and digital technologies, in general. Gregersen and Baccaglini-Frank (2022) also pointed out this issue when discussing that instrumented justification can become a matter of generating an example of a claim that is then taken as evidence. This is evident in the four pairs of students that justify as so in problem d: "No, because they are equal in (1,1) where $A1 = B$, and we cannot find a solution where $A2$ does not equal $A1$". As a result of the 'one technique only' strategy, the justification refers to the failure of that technique. As just elaborated upon, this challenge can point to issues related to several competencies; however, it is also notable that it is quite possible to challenge students to overcome this issue. In the cases of Em and Isa and Max and Sam, I approach them at a point in their problem-solving process where they are failing with their first chosen technique. In both cases, I indicate that this problem is solvable, which spurs them to try other techniques in manipulating the coordinate set, eventually leading them to a fruitful solution.

The potential of instrumentalizing components of the algebra view for justification

There is evidence that some, though few, students exercise their reasoning competency concerning algebraic properties in their work with the Equal Points task. To do so, the students start or continue a process of instrumentalizing the algebra view as an instrument for justification. This process entails epistemic mediation where the expressions in the coordinate set, along with imposed techniques, are related to the behaviour of the variable points. For this to be possible, they must, to some extent, have instrumentalized the graphic view in order to understand the relationship that is represented by the variable points, and have instrumentalized some techniques for solving the equality problems with variable points. Indeed, the Equal Point task provides students with the possibility to do so by providing a situation where they must make use of the algebra view in order to influence the objects in the graphic view. Three pairs of students in problem a and two pairs of students in problem b have final arguments that refer to algebraic properties, e.g. for problem b, "No, as *A2* is always one below *A1* because of the -1". However, as just debated, several challenges can stand in the way of the student going into a process of instrumentalizing the algebra view for justification. Yet, from the initial task to the second edition, more students attempted to solve problem d, which can be explained by the more direct formulation of the problem towards the specific coordinate. This shows that the reformulation supported more students in a process of instrumentalizing techniques in the algebra view for solving the given problem. However, as one might notice, the part of the problem formulation related to the justification was left the same. It might be possible in the same manner to direct the students towards using the algebra view for instrumented justification. Rather than just asking "*justify your answer*" or "*why/why not*", you could add, "*use information from the algebra view in your justification*" to direct students to focus on this as they attempt to justify their solutions.

Generalizability and characteristics of findings

To discuss the strengths and weaknesses of the results, I consider the generality of the findings as well as their theoretical characteristics (Schoenfeld, 2007).

First of all, the findings have contextual limitations that influence the study's generalizability. The age of the students and their prior experiences with GeoGebra and algebraic procedures influence the data. However, the potentials and challenges identified in the discussion can inform others attempting to support a similar age group of students, and thus has some predictive characteristics.

Secondly, the fact that some, though few, pairs of students exercise their reasoning competency, indicates that variable points as representations of algebraic properties of variables in expressions in GeoGebra can provide a context for students to exercise their reasoning competencies. Yet, for this to be possible, students need the necessary support and guidance for their instrumentalization of the tools in the algebra view for both problem solving and justification. Some support measures are implanted in the second edition of the Equal Point task, and in the discussion, I point towards suggestions for how these can be met in future task editions. These suggestions do hold some generality, e.g. guiding students towards specific components of the tool in the problem description. Here, the detailed description of the evolution of the task provides rigour (Schoenfeld, 2007) to the study, contextualizing how and why such measures are necessary.

Finally, the analysis provides an attempt at how to theoretically approach students' exercises of reasoning competency when working with digital environments that contribute to the general discussion about the influence of digital tools on students' mathematical competences (Geraniou & Jankvist, 2019).

Conclusion

In this paper, I described the evolution from the initial task to the second Equal Points task to gain insight into students' exercise of reasoning competency as they use tools from the algebra view in GeoGebra. Ultimately, I point to the potentials and challenges for the students' exercise of their reasoning competency. I do so by initially taking a global view of the students' justifications and the nature of their arguments, which in the initial task, act as a base to point out issues as well as the unfulfilled potential of problem d, as only one pair, Isa and Em, attempt to solve this problem. The potential of problem d is unveiled through prior results reported from an analysis of the pair of students along with a complimentary analysis of their instrumentation that demonstrates how the instrumentation of the students should be guided towards editing the coordinates of the variable point in the algebra view. In the second edition of the Equal Points task, the global analysis creates an overview of the nature of the students' final arguments, and along with a global analysis of the used techniques, I present different groups of students with informative cases (Flyvbjerg, 2006) to understand the challenges the students experience. The discussion shows how arguments of the phenomenological nature that consider the graphic properties of the variable points can be considered a stepping stone towards understanding the algebraic properties if the students can identify the relevant concepts at play. On the other hand, for the students who struggle with identifying relevant properties in the graphic view, the algebra view becomes an obstacle as it adds complexity that they do not yet understand.

Students' symbol and formalism competency also influence their ability to draw on the potentials of the algebra view to exercise their reasoning competency. Different misunderstandings or lack of knowledge about algebraic rules and procedures can cause faulty answers, e.g. knowing that $1s=s$, and if a student's instrumentation is not aimed towards justifying the possibility to discover, their fault is missed.

A challenge is related to the students' problem handling competency. A majority of the students stick to one strategy in the problem-solving process, standing in the way of them reaching a correct solution. Even when they can understand why their solution does not work, they do not attempt another. The failure of the technique influenced the nature of the students' justification, as the failure becomes evidence in their argument rather than the concepts at play in the task. It then becomes a challenge for the students to exercise their reasoning competency.

The potential for students' exercise of reasoning competency by tools in the algebra view lies in the structure of representations between the graphic view and the algebra view, which give students phenomenological impressions from which they can conjecture about the algebraic properties they have just acted on to change the behaviour of a variable point. According to the instrumental approach to mathematics education, the goal of activity is a component that directs the use of the tool. For students to exercise their reasoning competency by using tools in the algebra view, they must start a process of instrumentalizing techniques for justification, which also entails epistemic mediation. For this to be possible, they must already have instrumentalized the graphic view for justification to understand the relationship between the variable points and the coordinate sets in the algebra view.

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Appendix: Transcription of Isa and Em solving Equel points task d

Notes: The pair have constructed C= (2,s). The trace function is on for all points. Isa controls the computer. R is the researcher.

1.	[33:19]	Isa	Okay then, can *C* be equal to *B*? [observes GeoGebra]
2.	[33:24]	Em	Collide, colide, colide – no. [while observing the screen]
3.	[33:28]	Isa	No, they cannot be equal to each other. Because these [points at *A* then *B* with arrow tool], they can be equal to each other.
4.	[33:34]	Isa	But *C* can't.
5.	[33:35]	Em	No.
6.	[33:35]	Isa	That is because *C*, *C* is to slow.
7.	[33:39]	Em	It is too far away.
8.	[ee:42]	Isa	Yes.
9.	[33:44]	Em	It is too far away
10.	[33:45]	Isa	At least the *C* we have made can not.
11.	[33:48]	Em	Maybe another *C* can
12.	[33:50]	Isa	[Types in task sheet: " no it can not "]
13.	[34:01]	Isa	Okay, justify your answer [reads from task]
14.	[34:07]	Em	But there will probably be some that can, if they are further away. If they move like this, and *B* is moving here, then it must collide here. Then it's something like that we would have here. [Gestures in the air, but the view on camera are blocked by Isa]
15.	[34:17]	Isa	Yes, because *A* and *B* will always
16.	[34:20]	Isa Em	always collide
17.	[34:22]	Em	So this [*C*]should be at the same place, at the same time, so it's not possible.
18.	[34:25]	Isa	Sad –hm- Okay.. ehm… Wait a minute [Looks a GeoGebra]
19.	[34:37]	Isa	If we do like this. [Stops animation and Changes *C* to (2,2*s*) and starts animation]
20.	[35:08]	Isa	Observers animation
21.	[35:14]	Isa	Okay, so no. Why does it not? [click back arrow, and *C* returns to *C* = (2,*s*)]
22.	……		**A section where they justify for why *A* = *C* is not possible has been cut out.**
23.	[38:06]	Isa	Hmm, let's see
24.	[38:14]	Em	they will never collide
25.	[38:25]	Isa	they will never, ehm, it can never be *C* equal to B because *C* is to show [Turns on animation]
26.	[38:26]	Em	Because *C* moves parallel to *A*.
27.	[38:29]	Isa	Yes.
28.	[38:30]	Em	And when *A* and *B* collide, then *C* is next to it. All the time, so it will never be able to get there, unless…
29.	[38:39]	Isa	What?
30.	[38:40]	Em	No not unless. It just can't
31.	[38:45]	R	So what are you working on here?
32.	[38:48]	Isa	We are working on the *C* equal to *B*
33.	[38:51]	R	Isn't that is a tough one
34.	[38:53]	Em	But it can not!
35.	[38:54]	R	It can not?
36.	[38:55]	Em	no
37.	[38:56]	Isa	We can not get to do it
38.	[38:57]	Em	Because, when *A* and *C* are parallel , then *C* are to fare away, and cannot collide with the others.
39.	[39:08]	R	Yes, so if we were to move *C*? Try just the point on the screen.
40.	[39:10]	Em	Yes.
41.	[39:11]	R	You might want to stop the animation where A and *B* collide.
42.	[39:17]	R	See there they collide [everybody observes the animation]
43.	[39:19]	Em	Yes
44.	[39:20]	R	So if we were to move *C*
45.	[39:20]	Isa	(stops the Animation, and moves slider to *s*=1,05)
46.	[39:21]	Em	Hmm.
47.	[39:22]	R	Where would you place it, so they would collide just after *A* and *B*?
48.	[39:28]	Em	Ehm.
49.	[39:29]	R	Alright, so it could either be the same as *A*, but that is a little boring, we already know that one.
50.	[39:32]	Isa	Yes

51.	[39:]32	R	So if you were to move *C*, where would you move it to?
52.	[39:38]	Isa	Ehmmm
53.	[39:40]	R	You can also try and move it so that *B* collide with the trace of *C*
54.	[39:47]	Isa	[Adjusts s, accordingly, struggles a little to get the exact placement with slider]
55.	[40:03]	Em	right there Isa. [points at the crossing of *B* and *C*'s trace]
56.	[40:04]	R	Yes
57.	[40:15]	Isa	Ugh
58.	[40:15]	R	So where would you like *C* to be?
59.	[401i]	Isa	There [points at the crossing of *B* and *C*'s trace, *s* =1,8]
60.	[30:19]	R	Yes, so how can you get it there?
61.	[40:21]	Isa	Ehm
62.	[40:22]	R	What do you need to change for *C* to be there?
63.	[40:27]	Isa	We could…
64.	[40:35]	R	All right, I'll give you a couple of hints, then see what you can do with that. Consider, which of the coordinate values do you need to change? Is it the *x*- or the y-value? And how much do you need to change it?
65.	[40:46]	Isa	Okay
66.	[40:47]	R	Yes? [leaves]
67.	[40:49]	Isa	so, let's see. Look
68.	[40:51]	Em	Wait a minute. Must it collide with A at the same time?
69.	[40:58]	Isa	No not at the same time, it just needs to be parallel with *A*.
70.	[41:00]	Em	Okay, okay.
71.	[41:01]	Isa	And the lines don't need to have the same length.
72.	[41:04]	Em	Yes
73.	41:04]	Isa	[Isa clicks to edit the y-value of *C* – this also makes the trace disappear in the graphic view]
74.	[41:10]	Em	Can we do like this, and then we need to move *C* down there.
75.	[41:13]	Isa	Yes but how do we do that?
76.	[41:17]	Em	Right now it starts at two , so it starts there. Can we get to start further down?
77.	[41:23]	Isa	Oh yeah it starts here.
78.	[41:25]	Em	Can you get it to start at minus one?
79.	[41:31]	Isa	[Changes *C* from (2, s) to (-1,s) , starts animation.]
80.		Em & Isa	Both observes screen
81.	[41:37]	Em	Wait, they might collide. A still collides, so no
82.	[41:43]	Isa	No, but we need to change…
83.		Isa	Stops animation at s= -0,5
84.	[41:48]	Em	So we get it to start a little further down, then it might do like this. And A then, something, it will be before.
85.	[41:54]	Isa	Yes, but it is *s* we have to change.
86.	[41:58]	Em	Is it *s* we have to change then?
87.	[41:59]	Isa	Yes, can we do like this that?
88.			Types C = (2,0,5s) and starts animation
89.	[42:12]	Isa	Ugh
90.	[42:14]	Em	Ugh, where did *C* go?
91.	[42:19]	Isa	*C* is gone. Oh, there it is, what is it doing (*C* appears in 0,2 when s =< 0)
92.	[42:27]	Em	What?!
93.	[42:29]	Isa	Oh, it's because you can write it like that. Corrects C to *C* = (2,0.5*s*). Starts animation.
94.	[42:42:46]	Em	It is still moving parallel with *A*, Isa
95.	[42:47]	Isa	Yes, it is supposed to do that.
96.	[42:50]	Em	*B* and *A* still collides at the same time!
97.	[42:53]	Isa	Yes, but *C* is a little behind, *C* is half the time behind always, okay okay.
98.	[43:09]	Em	So, can do it!
99.	[43:10]	Isa	Yes!
100.	[43:11]	Em	Oh, so it was just a little too fast.
101.	[43:18]	Isa	Yes, so can *B* be equel to *C*? yes it can.
102.	[43:24]	Em	Good
103.	[43:25]	Isa	Okay, then we have to explain it.
104.	[43:31]	Em	First, we must show when. Reads from task "if yes, when?"
105.	[43:35]	Isa	Ehm, that is when s equals.. [observes animation]
106.	[43:44]	Isa	That is when s equals 2.

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Paper 5

Gregersen, R. M. (in review). Unveiling student's tool use and conceptual understanding in the prediction and justification of dynamic behaviors in *Digital Experiences in Mathematics Education*

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Analysing Instrumented Justification: Unveiling Student's Tool Use and Conceptual Understanding in the Prediction and Justification of Dynamic Behaviours

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Abstract

The study advances the instrumental approach to mathematics education (Drijvers et al., 2013; Trouche, 2003), aiming to elucidate the interplay between students' reasoning competency, conceptual knowledge and tool utilisation in dynamic digital geometry and algebra environments. The dynamic properties of these environments pose a nuanced predicament, as the outsourcing of translation between visual and algebraic representations raises concerns regarding students' conceptual development and reasoning competency. To mitigate this issue, a prediction task is proposed, focusing on the dynamic behaviour of variable points in GeoGebra. I introduce a comprehensive framework adapting Toulmin's argumentation model into the instrumental approach, emphasising processes of justification. This is complemented by the application of components of Vergnaud's (1998) scheme concerning generative and epistemic ways to approach how students' conceptual knowledge has played a part in these processes. Through a case study of a student pair solving a prediction task, I explore the links between instrumented justification, students' mathematical reasoning competency and conceptual understanding, and how students' use of GeoGebra tools is intertwined with their justification processes. The analysis reveals the intricate interplay between data production and interpretation, and it is grounded in inference drawn regarding students' implied theorems about concepts, dynamic behaviour and progression in terms of techniques. The results indicate that the progression of technique is driven by the experience of the inefficiency of techniques and artefacts related to the goal of justification. Essentially, the framework links students' reasoning competency to their use of tools and conceptual knowledge, as well as demonstrates that predicting dynamic behaviour can enhance knowledge-based justification.

Keywords Justification · Instrumental genesis · Reasoning competency · Dynamic geometry and algebra environment · *GeoGebra* · Variable · Prediction · Lower secondary education

Extended author information available on the last page of the article

Introduction

Today, most educational dynamic geometry environments (DGEs) allow the symbolic manipulation of geometric constructions and graphic representations. *GeoGebra* stands out as a DGE that fully integrates the traditional features of a DGE with the algebraic features of computer algebra systems, which is why *GeoGebra* can be used as a dynamic geometry and algebra environment (DGAE) with dynamic multi-representations (Hohenwarter & Jones, 2007). Dynamic properties increase the ability to examine mathematical concepts and relationships (e.g. Baccaglini-Frank et al., 2013; Nagle & Moore-Russo, 2013; Olive et al., 2010) and improve 'the reasoning, understanding, and conceptualization of mathematical objects' (Villa-Ochoa & Suárez-Téllez, 2021, p. 5). Dynamic behaviour presents a dilemma, as, on the one hand, it allows students to create and transform graphic representation through algebraic notation, which can otherwise be a challenging task, making algebraic manipulation and transformation more accessible to students (Hohenwarter & Jones, 2007), and, on the other hand, the outsourcing of translation between representations can be problematic.

It is widely acknowledged in mathematics education research that handling the various representations of a concept, including its associated processes and objects, plays a significant role in mathematical reasoning and concept development. For example, Sfard (1991) argues that shifting between representations of the same object is a necessary step toward reification, and Duval (2006) believes that being able to access and translate between different mathematical representations is crucial to all mathematical understanding and activity. Furthermore, Duval worries that the outsourcing of the translation of representations deprives students of an awareness of the one-to-one mapping between graphic visual values and algebraic terms.

Pedersen et al. (2021) suggest that, to address such issues, task designers could require students to predict changes in these representations when using digital technology. Moreover, students' justifications tend to rely on empirical knowledge (Harel & Sowder, 2007) or phenomenological evidence (Baccaglini-Frank, 2019). This tendency is enhanced by the dynamic properties of environments, which allow students to interact and observe representations that appear as *real* virtual objects that can be experienced phenomenologically (Baccaglini-Frank, 2019; Leung & Chan, 2006). In general, predicting results and strategies supports the development of students' reasoning abilities (Kasmer & Kim, 2011; Miragliotta & Baccaglini-Frank, 2021), which is why prediction tasks may address both the translation of representations and students' phenomenological tendencies. Additionally, in this study, I hypothesise that predicting the dynamic behaviour of objects in a DGAE allows students to reason about algebraic properties based on their mathematical conceptual knowledge while still capitalising on the *realness* of virtual objects and their dynamic properties.

In Denmark's education system, the mastery of mathematics is considered mathematical competence including mathematical reasoning competency as described in the KOM framework (Niss & Højgaard, 2019) (KOM abbreviates

'Competencies and the Learning of Mathematics'). Research on the use of digital tools in mathematics education's interplay with the development of mathematical competencies are becoming increasingly common (e.g. Bach, 2022; Geraniou & Jankvist, 2019; Geraniou & Misfeldt, 2022; Højsted, 2021; Jankvist & Geraniou, 2022; Thomsen, 2022). Mathematical reasoning competency includes the spectrum of forms of mathematical reasoning across the scope of mathematical mastery, from early mathematics education to expert mathematicians. This study focuses on justification as a particular aspect of reasoning competency, as justification is predominant in everyday teaching in mathematics classes in lower secondary education (age 13–16). Moreover, justification has been given little attention in the research on reasoning in general (Stylianides & Stylianides, 2022).

To address how students use tools in conjunction with their mathematical competencies, prior studies (Bach, 2022; Geraniou & Jankvist, 2019; Thomsen, 2022) have drawn on the instrumental approach to mathematics education (IAME) (Drijvers et al., 2013; Trouche, 2003, 2004, 2005). This approach highlights how tools become instruments used to solve mathematical tasks, but it does not delve into the processes of justification. In a recent study by Gregersen and Baccaglini-Frank (2022), we examined how the processes described in the IAME approach can be analysed as a componence of justification processes by using an adapted version of Toulmin's (2003) argumentation model. However, we did not extensively elaborate on the significance of students' conceptual understanding in justification processes.

Additionally, the KOM framework (Niss & Højgaard, 2019) has no concepts with which to analyse students' knowledge and conception, but it does recognise them as ingredients in mathematical competencies as the exercise of any mathematical competency involves some subject matter. In order to expand upon the conceptual aspect of students' justification processes in conjunction with digital tools, Geraniou and Jankvist (2019) have taken initial steps by suggesting that schemes (Vergnaud, 1998) as the cognitive component of the IAME may be useful in articulating 'the role of conceptual knowledge in relation to the mathematical competency' (Geraniou & Jankvist, 2019, p. 41). If so, it might be possible to link students' conceptual knowledge and development, reasoning competency and tool use by analysing students' schemes. Accordingly, this study aims to explore how students use tools in a DGAE in justification processes when predicting dynamic behaviour from both a reasoning competency and a conceptual perspective.

The framework and context of the study are further elaborated below, after which the aim will be concretised into two research questions. The explorations are then conducted as a case study (Thomas, 2011b) of the justification process of a pair of students solving a prediction task embedded in a restricted *GeoGebra* environment. The case is analysed in three steps. First, the potentials and constraints (Trouche, 2005) of the relevant tools are considered, followed by an analysis of the students' justification process and tool use using an adapted Toulmin's model (Gregersen & Baccaglini-Frank, 2022). Finally, the students' process is analysed with regard to the components of the scheme (Vergnaud, 1998).

Theoretical Framework

Reasoning Competency, Arguments and Justification

The KOM framework defines a mathematical competency as 'someone's insightful readiness to act appropriately in response to *a specific sort* of mathematical *challenge* in given situations' (Niss & Højgaard, 2019, p. 14; *italics in original*). Out of eight distinct competencies, this study is confined to the reasoning competency. Students exercise reasoning competency when they analyse or produce mathematical arguments (Niss & Højgaard, 2019). This can consist of oral or written arguments in various forms, in this case, justification. An argument is a chain of statements linked by inference in support of mathematical claims or solutions to mathematical problems (Niss & Højgaard, 2011, 2019).

A person's competency is an evolving situated entity that is developed over time through active participation in mathematical situations. Competency development involves expanding the *degree of coverage*, *radius of action* and *technical level*. Coverage pertains to the different aspects of a competency, such as active participation in various forms of reasoning. The radius of action considers the diverse contexts in which the competency can be applied, spanning various domains and social situations. The technical level addresses the sophistication of concepts, theories and methods.

Justification is the process of supporting mathematical claims and choices when solving problems when students are asked to explain and warrant their answers concerning a given problem (Stylianides & Stylianides, 2022). In all mathematical reasoning, arguments are put forward to change the epistemic value (the degree of certainty) of a claim (Duval, 2007). The epistemic value can be considered from the perspective of the reasoner or the general mathematics community (Duval, 2007; Harel & Sowder, 2007; Jeannotte & Kieran, 2017; Knuth et al., 2019).

Considering an argument from a structural standpoint, Toulmin (2003) suggests a geometric structured model (see Fig. 1), considering what constitutes a valid argument from epistemological and psychological perspectives. Toulmin's argumentation model structures the fundamental components of an argument, including the claim, qualifier, data and warrant. A claim is a statement along with its epistemic value (i.e. qualifier). The qualifier expresses the probability of the claim (e.g. false,

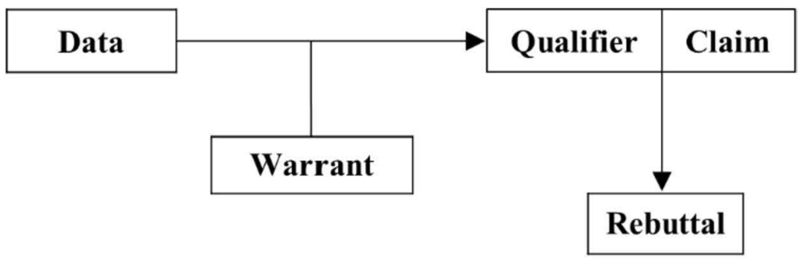


Fig. 1 Geometric structure of the elements of an argument (Toulmin, 2003)

possible, more possible or true) and is established based on evidence (i.e. data that supports the claim and the warrant, which connects the data to the claim). Finally, the rebuttal limits or counters the claim.

The Instrumental Approach to Mathematics Education

The IAME conceptualises how a tool becomes an instrument for solving mathematical tasks through the process of *instrumental genesis* (Artigue & Trouche, 2021; Trouche, 2003). The process comprises a subject (from here on, a student) and a material or non-material artefact. The student knows objects or concepts particular to the situation and the use of artefacts. The artefact *mediates* the students' actions on objects, which are influenced by the potentials and constraints of the artefact. As the student uses the artefact, it becomes a tool for a particular situation or task. This could be using a polygon tool (the artefact) in a DGE to construct a triangle (a task concerning an object). Instrumental genesis has a dual nature: *instrumentalisation* and *instrumentation*. Instrumentation is the constraints and possibilities imposed on the student's actions, while *instrumentalisation* is the student imposing a personal use. Over time, as the student uses the tool for similar situations, the process of instrumental genesis unfolds to develop an *instrument*. An *instrument* is a cognitive unit that consists of both *scheme* and *artefact*.

Schemes concern perceptual and gestural goal-oriented activities in 'the invariant organization of behaviour for a certain class of situations' (Vergnaud, 1997, p. 12). Schemes include a generative component: *rules-of-action*, which shape behaviour based on situational variables. The purpose of rules-of-action is not to be true, but to be effective. In addition, schemes include the conceptual components of the operational invariants: *theorems-in-action* about *concepts-in-action*. Theorems-in-action are often not explicitly stated, but rather *held-to-be-true* statements that, according to mathematical theory, can be true or false. Theorems-in-action provide insight into the world of objects: the concepts-in-action that can be relevant or irrelevant to the situation or task.

Moreover, *invariant* behaviour is relative as schemes are adapted by inference pertaining to contexts and circumstances (Pittalis & Drijvers, 2023; Vergnaud, 2009). This relativism reflects the instrumental genesis as an instrument develops over time. In the same manner, the stability in a scheme for a certain class of situation is reached over time. Consequently, the schemes of students using an unfamiliar artefact or solving an unfamiliar task will be less stable, rules-of-action may be ineffective and theorems-in-action may be wrong (Ahl & Helenius, 2018). Finally, Vergnaud (1998) emphasises the significance of possibilities of inference within schemes, acknowledging that inference and computation are inherent in any activity.

In the IAME, the conceptual aspect of the epistemic use of tools is prevalent (Shvarts et al., 2021). For example, Drijvers et al. (2013) consider the dualistic process of activity and conceptual knowledge as a *technique-scheme* duality in the instrumental genesis process. Epistemic use is most often explored through the identification of the invariant behaviour across users of an artefact and the conceptual understanding underlying different usage schemes. Alternatively, through the analysis of the operational invariants of students' developing schemes, Rezat (2021) explicates the students' rationales as expressions of knowledge. Such insight is indeed relevant in justification processes. Thus, by adopting a similar approach to that of Rezat, the scheme-technique duality can provide insight into the co-evolution both of conceptual development and of justification in the instrumental genesis process.

Rabardel argues that, 'it is necessary to analyze and understand what these activities are from the perspective of the users themselves' (2002, p. 31). Indeed, this is a prominent concern. Therefore, I take an inclusive approach to *technique*, one encompassing all gestures involved in student tool use, including hand movements, direct interactions with the artefact and verbal expressions of imagined activity. In addition, oral explanations of action can provide insight into concepts- and theorems-in-action (Rezat, 2021), since 'enunciation plays an essential part in the conceptualization process' (Vergnaud, 2009, p. 89).

Instrumented Justification

Traditionally, the IAME has been applied to analyse students' learning techniques when solving particular mathematical problems utilising a digital tool, such as determining the solutions to an equation utilising CAS (e.g. Artigue, 2002; Jupri et al., 2016). In this case, students utilise *GeoGebra* to predict the translation of symbols. Furthermore, students may present arguments in support of or against certain claims arising from their solution process. In order to capture such processes, Gregersen and Baccaglini-Frank (2022) introduced an analytical tool by reinterpreting Toulmin's model, in light of the *scheme-technique* duality, and termed the process *instrumented justification* (IJ). Based on this analytical tool, IJ is described as '*a process through which a student modifies the qualifier of one (or more related) claim(s) using techniques in a digital environment to generate and search for data and warrants constituting evidence for such claim(s)*' (Gregersen & Baccaglini-Frank, 2022, p. 135; *italics in original*). An elaboration of the analytical tool is provided below. Please refer to Fig. 2.

Toulmin's model is most often applied in mathematics education research to analyse a finalised argument or chains of sub-arguments. However, in the IJ analytical

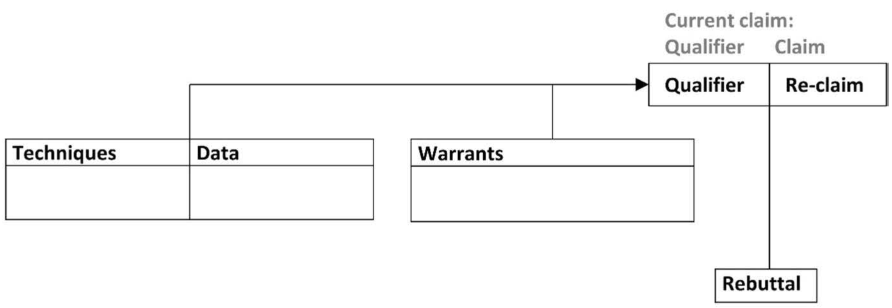


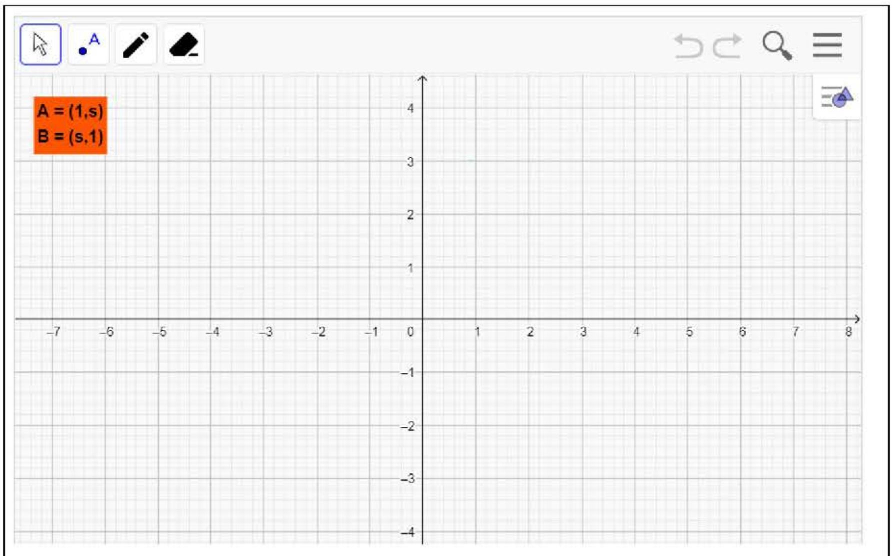
Fig. 2 Adaptation of Toulmin's model into an analytical tool for students' instrumented justification

tool, the unit of analysis is *the process* from a claim to a restatement of that claim, along with a change in the qualifier. The students generate data through techniques as evidence to support or refute the initial claim, and change the qualifier from 'possible' to 'more possible', 'less possible', 'true' or 'false'. The close connection between data and techniques appears in the analytical tool as connected frames correlating a technique to the data it produces. The schemes (Vergnaud, 1998) that direct and organise techniques generating data contain conceptual elements and rules that regulate actions that are seen as warrants that connect the data to the claim and can be inferred from students' techniques and verbal expressions (Rezat, 2021).

Figure 2 shows a generic diagram of the IJ analytical tool as an adaption of Toulmin's model. In continuous sub-processes, the first uttered claim, along with its qualifier, is noted in the top right corner in grey, so below is the re-claim with a new qualifier. Finally, the rebuttal consists of the limitations of the claim or counterarguments as in Toulmin's (2003) original model.

A Prediction Task to Situate an Instrumented Justification Process

The task presented in Fig. 3 originates from a sequence of tasks developed during my Ph.D. study. Collectively, the sequence explores a microworld of *variable points* in *GeoGebra*, which are ordered pairs containing a variable in more or less



Points can have a variable in the coordinates like these two points: A = (1, *s*) and B = (*s*, 1), where *s* is a variable.

1. Show in the co-ordinate system how you think points A and B move when *s* changes value. (You can use the tools in the toolbar; right-click and use the tools there.)
2. Justify your hypothesis – why do A and B move as you claim?

Fig. 3 Above is the restricted interface of a GeoGebra app for predicting the movements of points A and B, presented in the orange box: the available tools are 'move', 'point', 'pen' and 'erase', and below are the questions posed complicated algebraic expressions. The current task is the first prediction task in the sequence.

In question 1, the students are required to predict how the variable points will move in the co-ordinate plane, which the students must visualise and explain using a highly restricted interface in a *GeoGebra* app. The restrictions are enforced to prevent students from constructing the points by typing them into the algebra view. The tools available are ‘move’, ‘point’, ‘pen’ and ‘erase’. The student can turn on the trace of constructed points to trace any dragging of the point. In question 2, the students must justify their predictions in writing in a Word document. Research has demonstrated that prompting students to predict outcomes can encourage mathematical reasoning using previous knowledge (Kasmer & Kim, 2011; Lim et al., 2010). Some research studies students’ prediction as a product or the processes by which predictions emerge (Miragliotta & Baccaglini-Frank, 2021). I follow the latter approach, viewing student predictions as an *instrumented justification process* and treating predictions as claims about assumed dynamic behaviour in *GeoGebra*. Unlike traditional positions of prediction in mathematics education as a statement or conjecture anticipating either the solution to the problem or the strategy used to reach a solution (e.g. Boero, 2002; Kasmer & Kim, 2012; Palatnik & Dreyfus, 2019), the intention in this case is to leverage predictions and thus give ‘students the opportunity to defend or refute ideas’ (Kim & Kasmer, 2007, p. 298). Consequently, I consider the prediction task as a problem in itself, one that requires students to engage in IJ and operationalise their knowledge about variables and dynamic behaviour in *GeoGebra*.

Inspired by physics education, the prediction task involves anticipating outcomes in a way that is akin to experimental testing (Højsted & Mariotti, 2021; White & Gunstone, 1992). The dynamic behaviour of objects in *GeoGebra* creates the impression of real virtual objects, simulating movement and behaviour comparable to physical objects that can be experienced phenomenologically (Baccaglini-Frank, 2019; Leung & Chan, 2006). The prediction of such dynamic behaviour can be tested in the environment. In fact, although not part of the case presented, following the prediction task, the students are asked to test their predictions and consider the outcomes.

Student Knowledge of Variables and Dynamic Properties

In the prediction task, the concept of the variable is central. I will briefly elaborate on the concept of variables from a conceptual perspective and in relation to dynamic behaviour in DG(A)Es. The dual nature of concept formation and development in mathematics education research, which involves processes and objects, is widely recognised (Douady, 1991; Dubinsky, 1991; Noss et al., 2009; Sfard, 1991). For young students, concepts are initially tied to processes within specific numeric situations. Ideally, these concepts evolve into abstract objects, enabling the exploration of structures and relationships (Douady, 1991).

Concerning variables, Noss et al. emphasise that generalisation involves moving beyond the specific, recognising the structural properties, relationships and patterns that variables (and constant) represent. Introducing variables often marks students' first step into objectification, requiring them to perceive a letter as representing all values subject to the same computational manipulation as numeric values. In addition, Noss et al. (2012) problematise the static representations of paper-and-pencil tasks, arguing that such representations hinder students' progression in conceiving variables as 'the inevitably static (and therefore specific) figure that can be presented on paper is often problematic for students as it lacks a rationale for thinking generally' (p. 64).

The dynamic behaviour of objects in a DG(A)E reflects the process–object nature of concept formation as either a discrete collection of examples or continuous movement. Indeed, Miragliotta and Baccaglini-Frank (2021) describe that, in predicting dynamic objects, students may pin-point specific positions or envision, enact or imitate continuous movements. This also holds true for variable points, which can shift between positions in a co-ordinate plane or move along a trajectory. In a fully generalised conception, a variable point transcends dynamic properties, taking on the form of a line. The structural properties are then defined by the position of the trajectory or a line in relation to the co-ordinate system and other variable points.

Research Questions

After explaining the theoretical frameworks and laying out the task details, the research aim can be formulated as specific research questions:

> *In predicting the dynamic behaviour of variable points in a restricted GeoGebra environment, how can students' use of the point tool, trace function and pen tool interplay with their justification processes?*
>
> *To what extent can an analysis of the components of the student's scheme provide links between the process of instrumented justification, the student's mathematical reasoning competency and conceptual knowledge?*

Research Design and Method

Design of a Case Study

This study aims to conduct a fine-grained analysis of the conceptual evolution of students engaged in IJ to link reasoning competency, tool use and students' conceptual knowledge. To achieve this objective, the study is designed as a case study of a singular key-case (Thomas, 2011b) that will follow the IJ processes and the evolution of schemes. The case is presented as a temporal account (Thomas, 2011b), based on transcripts of students' utterances, descriptions and pictures of gestures, and screenshots of their computer screen that capture specific moments. The case provides the reader with contextual insights into the development of the use of particular tools for the prediction of dynamic behaviour, by exemplifying the intricate development of such a process (Thomas, 2011a). Furthermore, the case demonstrates the potential of the described task to provide a context for students' IJ justification processes.

The case consists of the IJ processes of the two students, Lev and Rio, who were collaborating on solving the prediction task. This pair was chosen because the students engaged in an IJ process characterised by the development of their tool use and their prediction. In particular, Lev was verbal about his assumptions throughout the process, making it possible to infer his warrants. Lev and Rio regularly use *GeoGebra* in mathematics class, though they mainly use the graphic view and geometric tools, including points and tracing. They have only used the algebra view to provide information on constructed objects. In the introductory part of the task sequence, students were introduced to constructing static points through the algebra view. In class, they have been introduced to the definition of a variable as an expression of all values, and procedures concerning variables in equations, functions and formulas. The pair was acquainted with plotting points on the co-ordinate system but had no experience with variable points before the experiment.

Regarding reasoning competency, the prediction task requires the students to expand their *radius of action*, as variable points are a new task. Concerning *coverage*, the students have experience with justification processes and prediction from their regular mathematics classes, mostly in the form of estimating the results of a computation. Concerning the technical level, Rio and Lev have no experience justifying variables as generalised numbers.

Data Collection

Data were collected from a class of 7th-grade students aged 13 to 14 during a classroom experiment. To encourage the students to express their assumptions and justifications, they were paired up and shared a computer to solve the task sequence. Additionally, the students were instructed to verbalise their thoughts and arguments while solving the tasks. OBS studio was used to capture the students' screen, voices, faces and upper bodies on video recordings during the experiment. In the classroom, the mathematics teacher and I were present to assist students with any questions or issues they may have had while completing the tasks. The video recordings were transcribed. The gestures were described and, if necessary, accompanied by images.

Data Analysis

The theoretical framework examines the case from three perspectives: the artefact, IJ and conceptual understanding. Each perspective also divides the analysis into three steps.

In step 1, following the IAME, I describe each of the three artefacts to have a clear understanding of the limitations and constraints of the available artefacts in terms of solving the task (Drijvers et al., 2013).

In step 2, I analyse the case using the analytical tool for IJ processes. The IJ model's components are identified using a theory-guided structured coding approach (Mayring, 2015). The categories were developed and revised in a cyclic process until applicable across students' IJ processes. Claims and re-claims are identified and constitute the analysis unit. A claim is an uttered tentative or final solution to the task, and a reclaim is an uttered statement similar to or referencing the claim along with an implied change in epistemic value. Any rebuttal is then identified. Techniques are then identified in the unit together with the corresponding data produced. Techniques can be performed or imagined in verbal expression. The data is the products of students' interactions with an artefact and their verbal interpretations of the data produced as evidence for or against the claim. The change in the qualifier of a claim is inferred from the students' actions and utterances, such as hesitation or continued search for data, which can indicate a lack of conviction in the claim's truth. Inferring warrants is a demanding process that requires interpreting how students' techniques and the data produced are relevant to the claim. Warrants are typically implicit, so formulating them is an explicating procedure and requires a narrow qualitative content analysis (Mayring, 2015). The formulation of warrants is revisited to ensure consistency with the source data in videos and transcripts.

Step 3 categorises warrants as either rules-in-action or theorems-in-action. Then, the order of appearance of theorems-in-action is used to infer the 'possibilities of inference' drawn between theorems-of-action concerning different concepts-in-action. The analysis method is further addressed and discussed in the step 3 of the 'Analysis' section.

Analysis

Step 1: Analysis of the Potentials and Constraints of Tools Specifically Regarding the Tasks

For the prediction task, students must anticipate the dynamic behaviour of points $A=(1, s)$ and $B=(s, 1)$ in the restricted interface. This requires translation from symbolic to graphic representation, which is typically outsourced. To aid in this process, students have access to the move tool, the point tool, the trace function and the pen tool. The point tool enables the placement of free points on the coordinate plane, allowing subsequent movement using the move tool. When using the point tool, students must assign numerical values to each point they place. For example, $A=(1, s)$ is expressed as a singular case. Multiple values of the variable

(s) can be depicted by plotting several points, shifting a single point or activating the trace function, which leaves a track of points where the point is dragged across the screen. However, tracing can be challenging when moving a free object as it is susceptible to cursor movements. The point tool and trace function are specifically designed for mathematical objects and properties. In contrast, the pen tool allows free drawing, requiring students to apply mathematical properties or functionality, such as the notation of values, sketching, plotting, tracing points or drawing lines.

Step 2: Analysis of the Instrumented Justification Processes

Here, I provide an analysis of Rio and Lev's IJ process. It is presented as two sub-processes: the first proces is captured in 1a (Figs. 4 and 5) and 1b (Figs. 6 and 7), and the second in subproces 2 (Figs. 8 and 9). The first sub-process is lengthy and is, thus, divided into excerpts 1a and 1b to make the analysis accessible. Each excerpt includes a transcript, screenshots of the students' work in *GeoGebra* and an analytical IJ model. Between sub-processes 1b and 2, a brief intermission is described, which is not considered significant for the overall process.

In the IJ analysis model, warrants are labelled according to concepts (**WV** for variable and **WP** for ordered pairs or points) and numbered according to appearance. As warrants reappear in the process, they are referred to by these abbreviations. If warrants are challenged of the two students, the warrant is assigned to the student expressing the specific warrant. In the transcripts, I am referred to as *Me*, gestures are described in square brackets and author notes are in italics. Rio is in control of the shared computer and mouse in all excerpts.

Instrumented Justification Sub-processes 1a

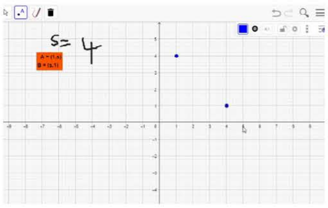
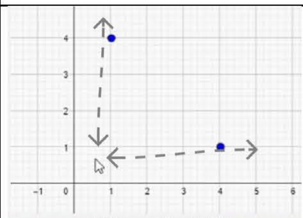
1	Me	You need to use these tools up here [gestures at tools in the tool bar] to show me how these points behave [gestures at points A and B in the orange box in *GeoGebra*].	
2	Rio	But shouldn't we then plot the points?	
3	Me	It's up to you how you want to do it.	
4	Rio	But how can we do that when we can't write that A equals 1? [*referring to the algebra view*]	
5	Me	Then you will have to try to show it. This one [gestures at pen tool] can draw, and this one makes points [gestures at point tool].	
6	Rio	So, should we just say that s = equals something?	
7	Me	That's up to you	
8	Rio	s equal [draws "s =" with pen tool]	
9	Lev	Ten, eleven, nine, no nine.	
10	Rio	We say four, like this. s equals four, right? [draws "s = 4" with pen tool]	
11	Lev	Boring.	
12	Rio	Then we can place a point that should be one [picks point tool and 'plots' by moving the cursor to one on the x-axis, then moving four up the y-axis].	
13	Lev	And then four.	
14	Rio	Here [places point A in (1, 4)].	
15	Rio	The other one is then s [counts four right and one up with the curser, then places point B in (4, 1)].	
16	Rio	Every time one shifts one place, the other one also shifts one place, so they keep moving further and further away from each other at a ninety-degree angle [moves curser from approx. (1, 1) towards A and then back to approx. (1, 1) towards B. Then, gestures with hands forming 90 degrees with wrists touching and moving his hands outwards in each direction].	 

Fig. 4 (**a**) The state of GeoGebra after excerpt 1; (**b**) illustrates the movement done with curser; (**c**) Rio's hand gesture

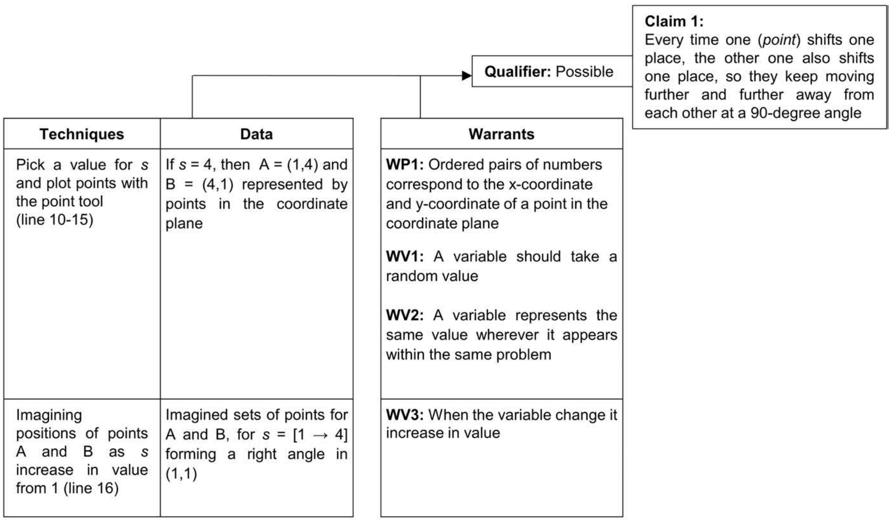


Fig. 5 Instrumented justification sub-process 1a through the lens of the analytical tool

Instrumented Justification Sub-process 1b

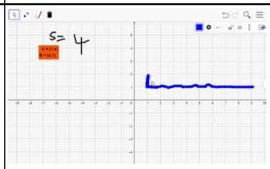
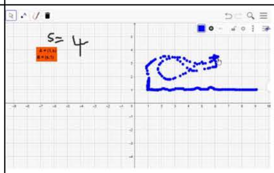
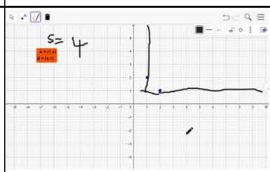
17	Rio	We can show how they start here by saying that *s* equals one [picks ‘move; tool, and moves both points to (1, 1)].	
18	Rio	Then, they are here, if s equals two [moves A: (1, 2) and B: (2, 1)].	
19	Rio	[Right clicks, turns on the trace for both points, then drags and traces B: (2, 1) – (9, 1).]	
20	Lev	That is very straight [*ironic about the trace*].	
21	Rio	Ah, no! [Drags and traces B along the exiting trace towards 0 but messes up the trace. Clicks “back arrow,” and the trace disappears. A: (2, 2) B: (9, 1).]	
22	Lev	What are you doing?	
23	Rio	[Drags and traces B: (1, 1) – (9, 1) then A: (1, 2) – (1, 1).] But it cannot go any further down, as we have decided that it [*s*] should be one.	
24	Lev	Yes, but it can go just as high or? Up to four, right?	
25	Rio	Ha! [Drags the point upwards along the y-axis but messes up the trace again.]	
26	Lev	It can go up to four.	
28	Rio	We need to do this, as we have to show it. [Drags and traces A: (1, 2) – (1, 5) and B: (2, 1) – (7.5, 1).]	
29	Lev	Just up to four.	
30	Rio	No! [Clicks on “back arrow”. The trace is gone. A: (1, 5) and B: (2, 1).]	
31	Rio	But, it is not up to four [turns off the trace function for both points].	
32	Lev	Yes, it is.	
33	Rio	No, *s* can just as well be other than four [drags A : (1, 5)→(1, 2)].	
34	Lev	You just wrote *s* equal four.	
35	Rio	Yes, yes, but if you change *s*, then *s* can shift [*beyond four*].	
36	Lev	But, you did not change *s*, did you?	
37	Rio	No [picks “pen” tool].	
38	Me	What are you boys trying to do now?	
39	Lev	To show how.	
40	Rio	So, it shifts like this. Every time they shift, they shift away from each other [draws line for A’s trajectory: (1, 1) – (1, edge of view) and B’s trajectory: (1, 1) – (edge of view, 1)].	
41	Lev	Then we need to explain it.	

Fig. 6 (**a**) Traces of points A and B ‘starting’ in (1, 1); (**b**) messing up the trace; (**c**) drawing trajectories of A and B limited by the ‘starting point’ in (1, 1)

Current claim:
Qualifier: Possible **Claim 1:** Every time one (*point*) shifts one place, the other one also shifts one place, so they keep moving further and further away from each other at a 90-degree angle

Qualifier: more possible

Re-claim 1: It shifts like this (*as drawn and gestured*). Every time they shift, they move away from each other

Rebuttal: The starting point is (1,1) which is also a limit

Technique	Data	Warrant
Move the plotted point to a new position corresponding to the change in the value of the variable (line 17-18)	If *s* = 1, then A = B = (1,1) represented by points in the co- ordinate system "they start out here"	**WV1, 2, 3** **WP2:** The shared position between points is a "starting point"
	If *s* = 2, then A = (1,2) and B = (2,1) represented by points in the co-ordinate system	
Activate the trace of points Drag a point to trace its movement Click the Back arrow when tracing goes wrong (line 19-36)	Trace of A: [(1,2);(1,1)] [(1,2);(1,5)] Trace of B: [(2,1); (9,1)] [(1, 1);(9,1)] [(2,1);(7.5,1)]	**WV4:** The variable should take on an interval of values **WP3:** An ordered pair with a variable corresponds to a set of points in the coordinate plane
	It (*the trace*) cannot go any further down, as we have decided that it (*s*) should be 1 It can go up to four No, *s* can just as well be other than four	**WP4:** A set of points is limited by the values we pick for the variable Lev: **WP4** Rio: **WV5:** The variable can increase infinitely but is limited by the "starting point"
Draw the trajectory of points with pen (line 40-41)	Drawn line for A's trajectory: [(1,1) →(1, edge of view)] and B's trajectory: [(1,1)→ (edge of view,1)]	(Both) **WV5** **WP5:** An ordered pair with a variable correspond to a set of points in the coordinate plane but are limited by the "starting point"

Fig. 7 Instrumented justification sub-process 1b through the lens of the analytical tool

Between Instrumented Justification Sub-processes 1b and 2

The students assert the claim by expressing that they have now shown how the points move and go on to the next task, justifying their prediction, which requires a written answer in a Word document. They begin by referencing **WV2** and restating the claim, but then Rio seems to realise that their claim is faulty and returns to *GeoGebra* to reconsider their answer to question 1. Then, Rio's following 'monologue' occurs, leading to a new claim (claim 2). As Lev is only observing in this excerpt, it is impossible to infer whether he shares the listed warrants. The written answer is not revisited.

Instrumented Justification Sub-process 2

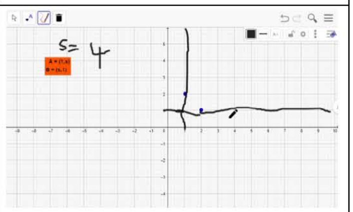
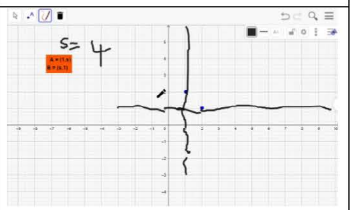
42	Rio	When one point changes and *s* becomes higher or lower, it moves the point back and forth, and then the other point moves in the opposite direction [*perpendicular*]. We know that they have one. This we said before also. So that is why they cannot move further than this [points curser at approx. (1, 1)].	
43	Rio	Or actually, they can. For example (1, 0), it is like this. They can actually move like this [prolongs both of the drawn lines through axes].	
44	Rio	We know that they have the number one, so they cannot – they cannot [circles the cursor at the intersections of the drawn lines and axes], but they can change in this direction [cursor moves back and forth on the trajectory of A] because *s* can be infinite.	
45	Rio	And actually, they can also move in this direction if *s* becomes negative [prolongs both of the drawn lines into negative co-ordinates for s: A's trajectory : (1, -3) – (1, edge of view), B's trajectory : (-3, 1) – (edge of view, 1)].	
46	Rio	But they will always be one away [from the axes], and then they can just change infinitely because of *s*.	

Fig. 8 (**a**) Extending trajectories to 0 on axes; (**b**) extending trajectories into negative numbers

Current claim:
Qualifier: more possible **Claim 1:** Every time one (*point*) moves one place, the other one also moves one place, so they keep moving further and further away from each other at a 90-degree angle

Qualifier: Possible*

Claim 2: they will always be 1 away (*from the axes*), and then they can just change infinitely because of *s*.

Technique	Data	Warrant
Imagining the trajectory of points A and B as *s* increase and decrease (line 42)	Imagined points moving back and forth on the drawn trajectories: A trajectory: [(1,1) →(1, edge of view)] and B's trajectory: [(1,1)→ (edge of view,1)] The starting point (1,1) is a limit for the trajectories	**WV6:** The variable can increase infinitely and decrease to the limit **WP2':** The shared position between points is a "starting point" defined by the common constant in the ordered pairs **WP5**
Draws the trajectory of points with pen (line 43)	Drawn line for A's trajectory: [(0,1) →(1, edge of view)] and B's trajectory: [(0,1)→ (edge of view,1)] The points seem to move through the starting point.	**WP6:** An ordered pair with a variable corresponds to a set of points in the coordinate plane, which can include the "starting point" **WV7:** The variable can increase infinitely and decrease to 0 beyond the limit
Imagining the trajectory of points A and B as *s* increase and decrease (line 44)	The starting point (1,1) is a limit for the trajectories Imagined points moving back and forth on the drawn trajectories	**WP2'** **WP5** **WV6**
Draws trajectory of points with pen (line 45)	Drawn line for A's trajectory: [(-3,1) →(1, edge of view)] and B's trajectory: [(-3,1)→ (edge of view,1)] There is no starting point	**WV8:** The variable can infinitely increase and decrease **WP7:** An ordered pair with a variable corresponds to an infinite set of points on a trajectory in the coordinate plane
Imagining the trajectory of points A and B as *s* increase and decrease (line 46)	Imagined points moving back and forth on infinite trajectories: A's trajectory: [(∞,1) → (1, ∞)] and B's trajectory: [(∞,1)→ (∞,1)]	**WP8:** A constant in an ordered pair corresponds to the constant distance to the corresponding axis of a trajectory

Fig. 9 Instrumented justification sub-process 2 through the lens of the analytical tool (as the student instrumented justification process continues into the testing step, the epistemic value of the claim is only possible)

Evolution of the Elements in the IJ Process

Rio and Lev put forward two different claims:

- Claim 1: Every time one (*point*) shifts one place, the other one also shifts one place, so they keep moving further and further away from each other at a 90-degree angle (with the rebuttal 'the starting point is (1, 1) which is also a limit').

- Claim 2: They will always be one away (*from the axes*), and then they can just change infinitely because of s.

When claim 2 is presented, it causes claim 1 to become less credible and ultimately be proven false. The two claims are not contradictory; instead, the second claim extends the first. The patterns of the trajectories are consistent in being perpendicular along the axes. However, the description evolves from only having positive directions in the first claim to having both positive and negative directions. Consequently, the limit is refuted. In sub-process 1a, claim 1 is based solely on data with $s = 4$ and the corresponding positions of points A and B. Rio imagines how the points will move from the starting point of (1, 1). Although Rio does not explicitly state why this is the starting point, it is later labelled as such in 1b and can be inferred to **WP2**. In sub-process 2, line 42, Rio explains that the constant within the ordered pairs defines the starting point. This understanding may have already existed in 1a. Therefore, **WP2** can be specified as **WP2'**.

I now return to sub-process 1a. The techniques encompass plotting points with the point tool corresponding to a chosen value of the variable and imagining the position of points. In sub-process 1a, the warrants **WV1, 2, 3** and **WP1** are inferred. In sub-process 1b, Rio produces data to support claim 1 by moving the points to different positions corresponding to other variable values. He then changes his technique and traces through the dragging of points to generate sets for each point. The shift in technique also evolves **WV1** into **WV4** and **WP1** into **WP3**. The tracing spurs a discussion between the two students about whether the initially chosen values of the variable (four and one) are also the limits of the trace. Though struggling with the trace function, Rio realises that the limit of four is unjustified, as he can continue the trace for higher values of s, evolving the warrant **WV4** into **WV5**. At this point, Rio changes the technique again as he continuously struggles with tracing. Instead, he draws the trajectories with the pen tool from the 'starting point' in positive directions to the edge of the graphic view. The limit of one is maintained, which can be explained by **WP2'**, though this has not yet been expressed, and **WP3** evolves into **WP5**. I will address Rio's thinking further in the coming analysis of the development of the components of schemes.

In sub-process 2, Rio recalls that the variable can both increase and decrease in value, which is inferred to **WV6**. The realisation seems ultimately to unravel the issues regarding the starting point, as the limit is moved to 0 and then infinitely into negative numbers. This process is expressed in warrants **WP6**, **WV7**, **WV8** and **WP7**. Each relates to generating new data as the trajectories are elongated. This process resolves the issues concerning the limit of the variable, and claim 2 is conceived of, in which s is infinite, and the points can 'move' infinitely on the trajectories. In line 44, Rio struggles to discard **WP2**, even though it conflicts with **WV6** and **WP6**. However, ultimately, Rio reinterprets the constants in the ordered pairs as the trajectories' distance from the axis, and **WP2** evolves into **WP8**.

Step 3: Warrants as Windows on the Components of Students' Schemes

I now turn to the evolution of warrants in the IJ process, by considering the components of the scheme (Vergnaud, 1997, 1998) as the students' progress in their IJ process. In the excerpts, Rio is both the active user of the tools and the most articulate. Consequently, most inferred warrants can be connected to his schemes alone. In sub-process 1a, Lev does challenge Rio's justification about the limit of four, which shows us a little about Lev's warrant at that specific point in the process. We cannot know the extent to which Lev assimilates his warrants according to Rio's justification; we can only observe that Lev does not object any further. Thus, in the following analysis, I will only consider Rio.

Remember that schemes are goal-oriented concerning the task at hand (Vergnaud, 1997)—in this case, the goal is putting forward a *prediction* and *justifying* that prediction by changing the epistemic value. Such activity involves both rules-of-action and theorems-in-action about relevant concepts-in-action: variables and ordered pairs as points in the co-ordinate system. Thus, it is possible to elaborate on warrants as rules-of-action generating techniques relying on theorems-in-action about concepts.

Rules-of-Action

Recall that rules-of-action are implicit propositions concerning the appropriateness of actions for a particular situation (Vergnaud, 1997, 1998). Consequently, rules-of-action can be appropriate and efficient, or irrelevant or inefficient. Some inferred warrants (see Appendix 1) can be considered rules-of-action as they are mobilised into different techniques. The warrants **WV1**: 'A variable should take a random value' and **WP1**: 'Ordered pairs of numbers correspond to the x-co-ordinate and y-co-ordinate of a point in the coordinate plane' are mobilised as techniques for plotting points for randomly picked values of the variables, initially, by counting the distance from the axes to place points A and B in the co-ordinate plane and, then, as imagined points in the co-ordinate plane or by moving the points to new positions in the co-ordinate plane corresponding to other random values of the variable. However, these plotting techniques are deemed ineffective in sub-process 1b. Instead, the rules-of-action **WV4**: 'The variable should take an interval of values' and **WP3**: 'An ordered pair with a variable corresponds to a set of points in the co-ordinate plane' are mobilised as techniques for sketching trajectories. At first, this is done by tracing and moving a point, but, as the tracing is difficult to control, this is too ineffective. The rules-of-action are, however, still relevant and they are mobilised as drawing trajectories with the pen tool instead.

Inferences Drawn Between Theorems-in-Action About Concepts-in-Action

Now, let us turn to the *theorems-in-action* (Vergnaud, 1997, 1998). Recall that theorems-in-action are *held to be true* propositions about *concepts-in-action*. Clearly, concerning mathematical theory, theorems-in-action can be false, partly true or true. The warrants not already identified as rules-in-action are theorems-in-action (see

Appendix 1). In the students' IJ process, most theorems-in-action are false, or only partly true propositions, and are disregarded during the IJ process, starting from **WV3** until the students reach the true proportions of **WV8** and **WP7 + 8** in sub-process 2. However, in sub-process 1a, **WV2**: 'A variable represents the same value wherever it appears within the same problem' is true and undergoes no evolution. Rio mobilises this theorem-in-action to interpret the co-variance of the points as a pattern of perpendicular trajectories, which remains consistent throughout the IJ process.

To reach a justified prediction, Rio uses *inference possibilities* to infer properties about concepts-in-action. In the following analysis, I attempt to understand better Rio's IJ process by suggesting what inferences he has made. I do so by considering the order of appearance (see Appendix 1) of the warrants that are theorems-in-action in the IJ process as the line of thought. This approach has a weakness, in that the order of gestures and speech is not necessarily the order of thought. However, observing these actions is our only possible observation to understand how the students' schemes evolve in the process. I then chart the adopted *possibilities of inference* (Vergnaud, 1997, 1998), since a proposition about one concept is inferred into a proposition about another in support of either claim 1 or 2.

A) Inference chain warranting claim 1

- As **WV3**: When the variable changes, it increases in value.
- **WP2**: The shared position between points is a 'starting point'.
- (And could be that **WP4**: A set of points is limited by the values we pick for the variable?)
- No, so **WV5**: The variable can increase infinitely but is limited by the 'starting point'.
- **WP5**: An ordered pair with a variable corresponds to a set of points in the co-ordinate plane and is limited by the 'starting point'.

B) Inference chain warranting claim 1

- As **WV6**: The variable can increase infinitely and decrease to the limit.
- And **WP2**: The shared position between points is a 'starting point' defined by the common constant in the ordered pairs.
- It must be that **WP5**: An ordered pair with a variable corresponds to a set of points in the coordinate plane and is limited by the 'starting point'.

C) Inference chain warranting claim 2

- But **WP6**: An ordered pair with a variable corresponds to a set of points in the co-ordinate plane, which can include the 'starting point'.
- So, **WV7**: The variable can increase infinitely and decrease to zero beyond the limit.

D) Inference chain warranting claim 1

- But **WP2**: The shared position between points is a 'starting point' corresponding to the constant in the ordered pair.

- And **WP5**: An ordered pair with a variable corresponds to a set of points in the co-ordinate plane and is limited by the 'starting point'.
- So, **WV6**: The variable can increase infinitely and decrease to the limit.

E) Inference chain warranting claim 2

- As **WV8**: The variable can infinitely increase and decrease.
- So, **WP7**: An ordered pair with a variable corresponds to an infinite set of points on a trajectory in the co-ordinate plane.

The students' IJ process starts with the false theorem-in-action **WV3**, creating two issues that the students must resolve to reach claim 2: the direction of the movement and the limits/starting point. These two issues are two sides of the same coin. If we consider **WV3** and **WP2**, the inference could be along the lines of 'because the variable (only) increases in value, the variable must have a starting point'. As there is little other information provided by the task, the shared position of the points in (1, 1) (the intersection of the trajectories of points A and B) is interpreted as this starting point. It is also possible that the inference moves from identifying (1, 1) as the starting point to inferring the direction of the movement as only positive.

In both cases, how can we understand these false theorems-in-action? They may relate to which properties of the variable are relevant in this situation. The students will have encountered situations in which the variable is a placeholder for a value (e.g. in the context of formulas). In such situations, one must select relevant numeric information from a context to replace the variable, which is a rule-of-action. By mobilising such a rule-of-action, Rio attempts to select numeric information from the ordered pairs of points A and B, which are constants of one in this case. From that perspective, the theorems-in-action **WV3 + 5** and **WP2 + 4 + 5** combine the properties of the variable as a placeholder and the properties of the variable as a general number. Such a warrant has not been inferred in the IJ analysis and is speculative.

In inference chain B, **WV6** indicates a turning point. Rio recalls that the variable can increase and decrease, which is inconsistent with inference chain A. It seems that Rio struggles to accommodate his scheme through inference C–E, moving back and forth between justifying claim 1 and claim 2. In inference chain C, he infers properties from **WV6** to properties of the points in **WP6**, but still refers to **WP2** and **WP5**. In inference chain E, Rio realises the full extent of the variable as a generalised number in **WV8**, rejects any limits on the variable and reinterprets the constants in the ordered co-ordinates of points A and B.

Discussion

I first address the first research question by discussing the interplay between tool use and the justification process. I then address the second question via a discussion of the applied framework. Then, I consider the prediction task as the prediction of dynamic behaviour and, finally, I comment on the limitations of the study. The notion that instrumental genesis is goal-oriented is a core assumption of the IAME.

The findings show how the goal of justification results in a particular process of instrumental genesis. By extending the IAME with an analytical tool for IJ, with the goal of changing the epistemic value of claims (Duval, 2007) from the perspective of the students (Duval, 2007; Stylianides & Stylianides, 2022), we can see how the instrumental genesis unfolds through the production and interpretation of data. The change in epistemic value occurs through an interplay of producing data and interpretation through inference between the operational invariants. The inference allows the production of additional supportive or contradictory data. This cycle continues until the epistemic value is changed.

As argued by Shvarts et al. (2021) and many others, the educational value of using tools relies on the epistemic processes that allow for students' conceptual development. The conceptual element of IJ is considered through the warrants inferred and the analysis of operational invariants. Through such analysis, the case provides an example of the epistemic use of tools as the students' progress in the complexity of techniques used to produce data and in their conceptual understanding of the interpretation of data. In addition, the analysis of operational invariants shows that progression in conception emerges through inferential possibilities. What drives the development of instrumental genesis from one artefact to the next?

We know from Vergnaud (1997, 1998) that *rules-of-action* concern the appropriateness of actions for a task and can be efficient or inefficient. In this case, this is evident in the first progression, from placing points to tracing the trajectory of the points. However, inefficiency is relative to the students. The plotting of points can, for other students, be considered efficient. Moreover, the progression from tracing to drawing is not connected to a change in the *rules-of-action*. Rather, it is the artefact and technique that is inefficient. This drives Rio to try a different technique that more efficiently produces data and is coherent with the *rule-of-action*. From this, we can argue that the inefficiency of both *rules-of-action* and the constraints of an artefact can drive the development of instrumental genesis.

What is particular to an IJ process of predictions is that inefficiency is related to the production of data to represent or contradict a prediction and the goal of changing the qualifier. Changes in technique, for example, from discrete to continuous dynamic movements, and artefact produce new types of data. The fact that inefficient *rules-of-action* and techniques advance instrumental genesis and the IJ process supports the idea of a scheme/technique duality, as proposed by Drijvers et al. (2013). In addition to compliance with IAME, this also shows that the constraints and possibilities of the artefacts influence the process of instrumental genesis and, consequently, IJ.

The IJ analysis tool makes a significant contribution to our understanding of the epistemic use of artefacts, representing a step forward in comprehending students' justification processes. Notably, it brings into focus the students' utilisation of tools with an orientation toward the production of data. Furthermore, the model underscores the importance of discerning how students interpret the generated data. In essence, the tool emphasises the student's perspective on instrument use by inferring warrants. These warrants play a pivotal role in interpreting data in alignment with a claim. This nuanced approach enables us to deliberate on the co-evolution of students' tool use, considering both the evolution of techniques and the evolution of justifications. In doing so, the IJ analysis tool provides a valuable framework for delving into the intricate dynamics of students' interactions with tools in justification processes.

The IJ tool links student's reasoning competency to the use of artefacts. In the concrete case, we see that students broaden their radius-of-action by engaging in IJ processes, which also reflects the technical dimension of students' competency, as they progress in terms of the complexity of techniques. The analysis of scheme elements shows that such a progression goes hand-in-hand with the conceptual development that emerges from the inferences drawn between operational invariants. This analytical step enables us to contemplate how the evolution of concepts is intricately linked to the exercise of reasoning competency. In this way, the additional analysis of scheme elements is a valuable tool for illuminating the nuanced interplay between students' reasoning competency and conceptual development when using tools.

For the students, the prediction task has a familiar theoretical component of ordered pairs and points in the co-ordinate system, with which they have several years of procedural experience. It also has a less familiar component, because the students have no experience with operationalising variables in this context. This balance between the familiar and unfamiliar allows students to use the co-ordinate system to identify patterns in symbolic terms, as this can be from a procedural conception. In the introduction, it is hypothesised that predicting the dynamic behaviour of objects in a DGAE will allow students to reason about algebraic properties based on their mathematical and conceptual knowledge, while capitalising on virtual objects' realness via their dynamic properties. In this case, the students (or at least Rio) recognise the constant as the invariant pattern of perpendicular movement and the variable as the infinite movement of the points.

As Noss et al. (2012) maintain, such an inference would not be possible in a paper-and-pencil environment, as both the constant and the variable would be represented statically. Predicting movement, rather than asking for a translation of the variable points, allows students to capitalise on the dynamic behaviour and provides a context in which they can develop their theories-in-action about terms, and move on to the following task about variable points.

Nevertheless, one concern is issues of the translation of representation, such as the one-to-one mapping of terms, which Duval (2006) problematised in relation to dynamic environment. In this case, how do the students perceive A and B in the final prediction? Do students perceive A and B as particular points that move, or have they objectified A and B as structural patterned movement or, possibly, a hybrid of the particular and generalised? Such questions could be addressed by observing the students' progression in the prediction of other variable points.

Another issue we witnessed in the case was how phenomenological impressions (Baccaglini-Frank, 2019) can be a stumbling block to students' reasoning processes. This relates to the issue of a starting point. The students' experience of the physical world is that things that move begin moving while constrained by time and space. Consequently, the students misinterpreted the starting point from the common constant in symbolic terms in the variable point. However, through inference, the students overcome this misinterpretation, reaching an interpretation of the data and a prediction based on mathematically true *theorems-in-action*. This advances the hypothesis that prediction tasks can address students' tendencies toward phenomenological justification.

Some limitations of the results should be clarified. As the results are only based on one case, they are suggestive regarding the progression the students portray. Analysis across a wider set of cases will allow researchers to reveal the relationships between progression in tool use, issues that arises concerning the prediction of dynamic behaviour and student conceptual development in justification processes. Similarly, considering students' instrumental genesis during similar prediction tasks could reveal how invariant behaviour affects the prediction of dynamic behaviour of variable points.

Conclusion

This study offers an analytical tool that can increase our understanding of students' use of tools, in interplay with their reasoning competency, from a student-centred perspective within the IAME. Via instrumented justification, we can consider tool use as a particular use in justification processes, and as structuring such processes because of the production of data through techniques and the interpretation of data, as evidence, through warrants. In addition, an analysis of warrants as the generative and epistemic components of schemes (Vergnaud, 1998) provides insights into the progression of students' conceptual understanding as a result of inferences drawn between *theorems-in-action* about *concepts-in-action*, which co-evolve with the progression of techniques. In addition, the progression of instrumental genesis is driven by students' experience of the inefficiency of both *rules-of-action* and the constraints of the artefact pertaining to the goal of changing the epistemic status of a claim.

Inefficiency drives students to progress to other techniques and artefacts, ultimately advancing their instrumental genesis. This result aligns with the scheme/technique duality, which is a component of instrumental genesis as proposed by Drijvers et al. (2013). Such a result can inspire task design that intentionally provides students with ineffective tools, with a view toward progression to more advanced tools and the conceptual development that comes with this. Altogether, the proposed framework links students' progression in the radius of action in their reasoning competency and the use of tools to inferences drawn between *theorems-in-action*. Furthermore, the prediction task provides context for students engaged in IJ.

The prediction task is particularly valuable, as the prediction of dynamic behaviour reveals properties of the variable, as a concept, in this case infinity. In addition, the students must interpret both constant and variant terms, explicating their structural properties in the very simple representational form of points moving in a co-ordinate system. Such tasks may be useful in developing a structural conception of the variable. Within a more general perspective, the prediction of dynamic behaviour that requires the translation of representations can challenge students' phenomenological impressions of dynamic behaviour and help them move toward a theoretically grounded justification.

Appendix

Appendix 1: Warrants

Appearing order of warrants

In IJ sub-process 1a:

WP1: Ordered pairs of numbers correspond to the x-co-ordinate and y-co-ordinate of a point in the co-ordinate plane

WV1: A variable should take a random value

WV2: A variable represents the same value wherever it appears within the same problem

WV3: When the variable changes, it increases in value

In IJ sub-process 1b:

WV1, 2, 3: reappearing

WP2: The shared position between points is a 'starting point'

WV4: The variable should take on an interval of values

WP3: An ordered pair with a variable corresponds to a set of points in the co-ordinate plane

(Lev) **WP4**: A set of points is limited by the values we pick for the variable

(Rio, then both) **WV5**: The variable can increase infinitely but is limited by the 'starting point'

WP5: An ordered pair with a variable corresponds to a set of points in the coordinate plane and is limited by the 'starting point'

In IJ sub-process 2:

WV6: The variable can increase infinitely and decrease to the limit

WP2': The shared position between points is a 'starting point' defined by the common constant in the ordered pairs

WP5: reappearing

WP6: An ordered pair with a variable corresponds to a set of points in the coordinate plane, which can include the 'starting point'

WV7: The variable can increase infinitely and decrease to zero beyond the limit

WP2', WP5, WV6: reappearing

WV8: The variable can infinitely increase and decrease

WP7: An ordered pair with a variable corresponds to an infinite set of points on a trajectory in the co-ordinate plane

WP8: A constant in an ordered pair corresponds to the constant distance to the corresponding axis of a trajectory

Rules-in-action mobilised into a technique:

WV1 and **WP1**: Plotting points for randomly picked values of the variable

WV4 and **WP3**: Tracing or drawing trajectories of points

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Paper 6

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On the notion of ”background and foreground” in networking of theories

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In this paper, we report on a finding in an ongoing literature review on *Networking of theories*. As theories are the focus of networking practices, discussion of what is meant by theory is an ongoing debate. In our reading of these discussions, we experience a discrepancy in the use of the notion of background theories and foreground theories, which can be related to an absolute or a relative understanding of these notions. We account for this discrepancy and discuss potential consequences of each perspective to argue that a new notion "framing theories" or a distinction between "background theory inside mathematics education" and "background theory outside mathematics education" may accommodate these consequences.

The term ”networking of theories” stems from the thematic working group (TWG) on theoretical perspectives and approaches in mathematics education research (MER) at the *Congress of European Research in Mathematics Education* (Kidron et al., 2018). The group confronts the issue of the diversity of theories in mathematics education, and claims that ”theoretical approaches can *only* become fruitful *if* connections between them are actively established” (Bikner-Ahsbahs et al., 2014, p. 8). Taking this stance, the group has embarked on the challenge of how to establish connections between theories by developing ”networking of theories” as a research practice. Several important questions and issues have been discussed over the years. Kidron and colleagues (2018) state the following examples: ”What are the aims of connecting theories? [...] To what extent does the networking depend on the theories that are considered?” (p. 257); ”To what extent do we share the same notion of theory? (p. 257); ”What are the different aims of networking?” (p. 258); ”What do researchers do when they use more than one theory? Do the different approaches use the same words with different meanings?” (p. 258). Such questions have been addressed in the

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literature on networking of theories, e.g. Bikner-Ahsbahs and Prediger (2006) the ZDM article ”Diversity of theories in mathematics education – How can we deal with it?”, the ZDM issue ”Comparing, combining, coordinating – networking strategies for connecting theoretical approaches” edited by Prediger et al. (2008), and not least in the recent book ”Networking of theories as a research practice in mathematics education” edited by Bikner-Ahsbahs and Prediger (2014). Surely, the potential answers must to some extent draw on a common notion of ”what theory is” – we return to this below. For now, we draw the attention to the observation that in the available literature on networking of theories, there are often references to the notion of background theories and foreground theories (to be explained in more depth below) – this often occurs with specific reference to Mason and Waywood (1996), who initially introduced the terms into MER. Our ongoing review, which so far encompasses 96 publications on networking of theories, reveals the observation that the use of these two terms in more recent literature do not necessarily align with the original description by Mason and Waywood. More precisely, although some theoretical perspectives are attributed the role of background theories; these are not necessarily used in the sense of Mason and Waywood. Hence, there is a discrepancy between the descriptions and the actual use. In this paper, we ask the question: *How are the notions background theories and foreground theories used in the literature on networking of theories?*

We do not provide a full account of the 96 publications due to the space limitations of this paper. Instead, we present and discuss our finding through two carefully selected illustrative cases, showing the discrepancy in the use of background theory. Before we get to these cases, we briefly discuss the notion of theory itself and explicate the original notion of background and foreground theories as defined by Mason and Waywood (1996).

What is ”theory” in mathematics education research?

In networking of theories, a minimum requirement must be that we can agree on what is and what is not *a theory*. The literature – not only in mathematics education – is rich on various attempts of coining what theory is. For the reader who is unfamiliar with this discussion, we provide a brief account in this section. The reason we do this is not to apply this in our further analyses, but rather as a general comment to the ongoing discussion on what a theory actually is, and not least what a theory must be described by in order to be networked with other theories. We shall consider *a theory* from the perspective of networking theories, not least, with reference to what has taken place in this literature.

Kidron et al. (2018) state that the questions of what a theory is and how theoretical frameworks shape MER ”came into play when comparing or just talking about theories is the heterogeneity of what is considered as a theoretical framework in MER and the consequent possible incommensurability of the investigations that are carried out in different theories" (p. 261). Radford (2008, p. 320) suggested that a theory is a way of producing understanding and ways of action based on a triplet PMQ:

> A system, P, of basic principles, which includes implicit views and explicit statements that delineate the frontier of what will be the universe of discourse and the adopted research perspective.
>
> A methodology, M, which includes techniques of data collection and data-interpretation as supported by P.
>
> A set, Q, of paradigmatic research questions (templates or schemas that generate specific questions as new interpretations arise or as the principles are deepened, expanded or modified).

Around the same time, Prediger et al. (2008) surveyed different notions of theory found in the literature. This led them to distinguish between static and dynamic notions of theory, eventually pleading for a dynamic understanding: "theories or theoretical approaches are constructions in the state of flux" and they "consist of a core, of empirical components, and its application area. The core includes basic foundations, assumptions and norms, which are taken for granted" (p. 169). Niss (2019), however, notes: "The fact that theories or theoretical approaches are in a state flux doesn't mean that the definitions of the concepts are as well". We agree with Niss (2019) that: "Anything called a theory (or theoretical framework, construct etc.) is a *theory of something*! I.e. it deals with certain sorts of *objects* and *phenomena*, as well as *terms* for these". Mason and Waywood (1996) define such *objects* as the "sorts of things that are studied, even if they are not perceived as 'things' in any material way" (p. 1058). From Radford's (2008) account, it is unclear where these objects reside, although several researchers in networking of theories seem to consider them as part of the principles (P).

Foreground and background theories

As mentioned in the introduction, Mason and Waywood's (1996) distinction between foreground and background theories is often referred to in the discussion of the concept of theory. In this section, we outline our interpretation of the distinction as a basis for further discussion. Mason and Waywood (1996) present theory as a "hypothesis, or possibility such as a concept that is not yet verified but that if true would explain certain facts or phenomena" (p. 1055). They define foreground theory as *explicit* hypothesising based on the process of asking and answering questions within mathematics education, because "[…] the foreground aim of most mathematics education research is to locate, precise and refine theories *in* mathematics education about what does and can happen within and without educational institutions" (Mason & Waywood, 1996, p. 1056).

Thus, from the process of questioning "things" within a local or specific area of mathematics education research gives rise to new theories in forms of explicit hypotheses about what is happening, or what can happen under certain circumstances. The foreground theories are generated *within* mathematics education and can have one or more of four different functions: descriptive; explanatory; predictive and informing practice. Conversely to foreground theory, Mason and Waywood define background theory as *implicit* hypothesising or as a belief that guides behaviour. They consider that "every act of teaching and of research can be seen as based on a theory *of* or *about* mathematics education" with reference to Thom (1976), who puts it as "all mathematical pedagogy, even if scarcely coherent, rest on a philosophy of mathematics" (quoted in Mason & Waywood, 1996, p. 1056). In this sense, the theory remains in the background and implies an implicit way of action or behaviour of the teacher or researcher, but is not used with an explicit aim. It is important to notice that a background theory does not become a foreground theory, just because the hypothesis becomes explicit. Mason and Waywood (1996) emphasise that as a researcher, it is important to be aware and explicit about one's own background theories and their implicit assumptions and hypotheses. They explain:

> Background theories encompass an object (aims and goals of the research, including what constitutes a researchable question [...]), objects (what sorts of things are studied, [...]), methods (how research is carried out, validated and applied), and situation (as perceived by the researcher), and provide a language for discussing these. The situation necessarily assumes, manifests, encompasses, and is constituted through a philosophic stance manifested in the discourse and in other practices. (p. 1058)

This implies that the activities of research, such as framing researchable questions, using an appropriate method, collecting data, using analytical tools and looking at results as well as the validation hereof, are all determined and constructed by the background theory. This is elaborated with examples of how theoretical positions such as post-modernism, phenomenology and different directions within constructivism stress different ways and methods to investigate sociological and psychological dimensions and phenomena in educational research. Hence, we understand background theory as the theory that affords the conditions for the structure of the research, but it is *not* a theory generated within mathematics education research (MER). In addition, MER draws on theories from domains such as psychology and sociology, and their philosophical positions as well as their methods (Mason & Waywood, 1996). Accordingly, we understand Mason and Waywood's (1996) explanation of background theories as theories establishing the view by which we look at mathematics education, for example critical theory, constructivism, social-constructivism, phenomenology or ethnology. It also follows that we understand their term of foreground theory as the theoretical constructs generated and developed by research in mathematics education that have explicit aims in forms of describing, explaining, predicting and/or informing specific situations, concepts and practices happening or possible to happen in the teaching and learning of mathematics.

As an example of the differences between foreground and background theories, we use Vergnaud's (2009) *Theory of conceptual fields* (TCF). As TCF is a theory developed in MER, specifically concerned with mathematical learning, it is a foreground theory. To consider the background theories of TCF, we must understand what theories precede TCF. As Vergnaud (2009) argues for his perception of schemes, he draws on Vygotsky's (1962) as well as Piaget's (1977) understanding of schemes. These two constructivist perspectives both have a broader scope on learning as they are developed outside of MER. Hence, we position them as the background theories of TCF.

A hermeneutic literature review

The following is a brief overview of our initial literature review on networking of theories. This review was conducted as a hermeneutic literature review. Due to very limited results in databases, a systematic literature review was not possible to conduct (Boell & Cecez-Kecmanovic, 2014). As a part of a hermeneutic process, the understanding of the literature is never final; a constant re-interpretation is taking place. We began by scanning CERME proceedings, relevant ZDM issues and books and reference lists for the relevant literature to expand our literature base. Furthermore, we did literature searches in MathEduc and ERIC, although this did not reveal many relevant sources. Only literature describing the practice of networking of theories in mathematics education were included in the final cohort. We described each relevant piece of literature in the following categories made our findings about background theories more explicit: 1) actual results; 2) how is networking of theories used and discussed; 3) what theories are being networked; 4) what strategies and methods are applied; and 5) perspectives with particular relevance to our overall project.

In our efforts to grasp the discussions of category 2, we compared the use of the notion of foreground and background theories in the literature on networking of theories to the original reference by Mason and Waywood (1996). Our two cases are carefully chosen to illustrate the result of this comparison: Each case utilises background theory explicitly, yet differently. But first, a further elaboration on the different uses of background theory in networking of theories.

Foreground and background theories in networking of theories

In relevant literature, the use of Mason and Waywood (1996) is widespread, both in paragraphs concerning theory and in discussions thereof. At CERME5, a communication problem within the field of MER was noticed: ”Researchers from different theoretical frameworks sometimes have difficulties to understand each other in depth because of their different backgrounds, languages and implicit assumptions” (Arzarello et al., 2007, p. 1618).

This quotation emphasises the need to understand the origin and background of theories as well as their implicit assumptions and hypotheses. According to Bikner-Ahsbahs and Prediger (2006), the distinction between background and foreground theories seems applicable when analysing theories and their functions in different phases of research. This could be the characterisation of foreground theories and their respective background theories. An example is: ”The theory of interest-dense situations is a foreground theory with a middle range scope (Mason and Waywood, 1996), situated in the background theoretical framework of interpretative research on teaching and learning” (Bikner-Ahsbahs & Halverscheid, 2014, p. 99).

According to Bikner-Ahsbahs and colleges (2014), the underlying theoretical assumptions must be explicit when networking theories. Bikner-Ahsbahs and Prediger (2006) point out that ”the background theory and its philosophical base are deeply interwoven” (p. 53). For instance, when taking a constructivist perspective, mathematics has a philosophical view on the construction of knowledge. Nevertheless, the use of foreground and background theories is regarded neither as a definite definition of theories, nor as an absolute categorisation of theories. This leads to a more relative use of background and foreground theories, than originally intended by Mason and Waywood (1996), e.g.: ”In contrast [to the absolute definition], the status of some parts of the theory can change from foreground to background theory or vice versa within the research process” (Bikner-Ahsbahs & Prediger, 2006, p. 54). We interpret this statement to mean that a theory is not only *of/about* MER or only *in* MER, but that a theory can act as either, depending on the situation. Bikner-Ahsbahs and colleges (2014) contribute to this meaning by referring to foreground and background theories as *relative distinctions*. Still, and despite the discussions of making background theories explicit, authors reporting on networking processes and results seldom explicate the distinction. Hence, the way these terms are used within research practices are less apparent that one might initially anticipate.

Examples on the different use of background theory

Our first case is an example of the relativism of the notions as presented in Bikner-Ahsbahs and Prediger (2006). Koichu (2013) describes the work of a colleague in which a selected framework is contrasted with another. The insights obtained in the contrasting process are used in a following process of unpacking a selected construct in the selected framework:

> To this end and consistently with the Bikner-Ahsbahs and Prediger's (2006) terminology, the former theory can be seen as a foreground one, and the latter – as a background one. On the other hand, they use the Hershkowitz et al.'s (2001) work as a background theory or as an overarching framework, in which their own foreground theory is embedded.
>
> (Koichu, 2013, p. 2841)

The relativism of the status of a given framework thus becomes apparent as something that emerges in particular situations in research activities expressing the relation between frameworks in use.

Our second case is an example of another use of the notion of background theory. First, Fetzer (2013) addresses a specific perspective, namely Latour's *Actor network theory* (ANT) as a background theory to understand objects in mathematics education: ”*Latour's approach is fascinating and irritating and provokes the research question, if and respectively how actor network theory can be a fruitful background theory to get a better understanding about the role objects play in mathematical learning processes*” (Fetzer, 2013, p. 2800, italics in original).

Using Latour's ANT, Fetzer (2013) presents an example in line with Mason and Waywood's (1996) distinction between foreground and background theories. Latour's ANT, as a theory outside of mathematics education research, is used as a background theory determining the researchers' definition of an object, the researchable objects, methods and situations. Similar utilisations are found in Bikner-Ahsbahs and Prediger (2014) and Bikner-Ahsbahs and Halverscheid (2014). This way of using the notion of background theory implies that it is a perspective outside of MER, which allows the researcher to understand mathematics education through a particular philosophical or epistemological stance.

To sum up, our literature review on networking of theories indicates that the original terms, as defined by Mason and Waywood (1996), have undergone further development. The use of the notion of background and foreground theories in the networking of theories literature now also encompasses a more relative definition of background theory, i.e. one focusing on the relations of theories within MER.

Coexistence of two notions of background theory

In the discussion of theories related to networking of theories, Bikner-Ahsbahs and Prediger (2014) suggest to take ”the notions of foreground and background theory as offering relative distinctions rather than an absolute classification, they can help to distinguish different views on theories (p. 6). This quotation clearly describes the development of the definitions of foreground and background theories. Hence, in line with the findings of our literature review, and as showcased by the two illustrative cases above (Fetzer, 2013, and Koichu, 2013), the relative and the absolute distinction of the foreground and background theories coexist in literature on networking of theories. Schoenfeld (2007) emphasizes a need for specificity of concepts in research, as loosely defined terms can produce variation in results. Looking at the absolute distinction of background theory, this satisfies Schoenfeld's criteria for specificity. However, what are the potential consequences of an absolute distinction of background and foreground theory? One consequence is that it causes a large number of foreground theories, because all theoretical developments and contributions generated inside MER are considered as such. Another consequence of the absolute distinction is an untended need for a notion that denotes the experienced distinctions between theories *inside* MER. Using Koichu (2013) as an example of Bikner-Ahsbah and colleges' (2014) relative use of the notions, theory *in* mathematics education has a similar role as a background theory. Hence, the use of foreground theory as a background theory seems to confuse the use of background theory, since *background theories inside mathematics education* and *background theories outside mathematics education* then coexist.

Moving to the relative distinction of background and foreground theories, also this might not withstand Schoenfeld's (2007) criteria for specificity. A first consequence of a relative distinction is a less clear definition of foreground and background theories. A second consequence is the existence of different utilizations of the notion of background theory. When different utilizations of background theory exist, a third consequence occurs: The importance of the background theories *outside* MER and its philosophical base might be indistinct. If researchers do not take their background theories *outside* MER into account, the implicit assumptions and hypotheses continue to be tacit.

Conclusion

Our study shows that both a relative and an absolute distinct of foreground and background theories exist in the literature of networking of theories. Koichu's (2013) uses the relative distinction when denoting the relation between theories or frameworks in use. Fetzer (2013) uses the absolute distinction when she considers the underlying beliefs or epistemological position that determines the researches' goals, aim, questions and objects. Considering both the absolute and the relative distinctions, the following consequences appear:

- Adhere to the *absolute distinction*: a need for a new notion distinguishing background theories emerges when networking.
- Adhere to the *relative distinction*: different utilizations of background theories appear.

The consequences of both reveal the need for distinguishing between foreground theories, background theories *inside* MER and background theories *outside* MER. In networking of theories, the relative distinction also builds on the changing relationship between the theories used in a research practice (Bikner-Ahsbahs & Prediger, 2006). This means that one theory may act as both foreground and background *inside* MER.

Looking at the consequences of an absolute and a relative distinction between foreground and background theories, these indicate the need for a new distinction/notion. We suggest that the *background theories inside mathematics education research* are referred to as *framing theories*. Looking at Koichu (2013), the new distinction informs and describes the different roles of foreground and background theories in networking. If the notion *framing theories* is applied, the importance of background theories *outside* MER arises and the implicit assumptions and hypotheses in background theories *outside* MER thus becomes clearer. The new notion is not needed to characterise Fetzer's (2013) networking practice and the distinction between foreground and background. However, given the use of background theory outside MER and foreground theory inside MER, the theories involved do not change between the two types in a networking practice. This means that the dynamic relationship between theories only exist between *framing theories* and foreground theories *inside* MER.

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APPENDIX

List of appendices:

A: Iteration 2 task sequences

B: Iteration 3 task sequence

C: Abbreviations

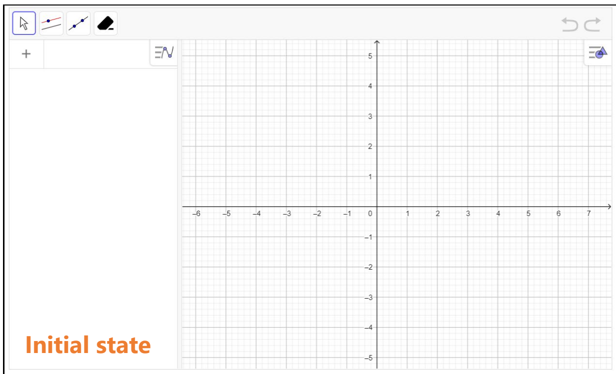
A: Iteration 2 task sequences

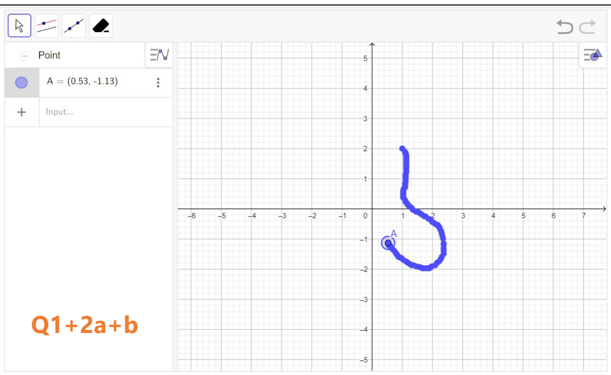
The fully online task sequence implemented in class A

https://www.geogebra.org/m/pbrvygs5 (In Danish)

Word document with tasks sequence + online GeoGebra apps English

Frist and introductory problem set





Q1

Read and do:

Points are constructed by defining their x and y coordinate. Points are always named with a capital letter.

Try it yourself, enter the following: A = (1,2)

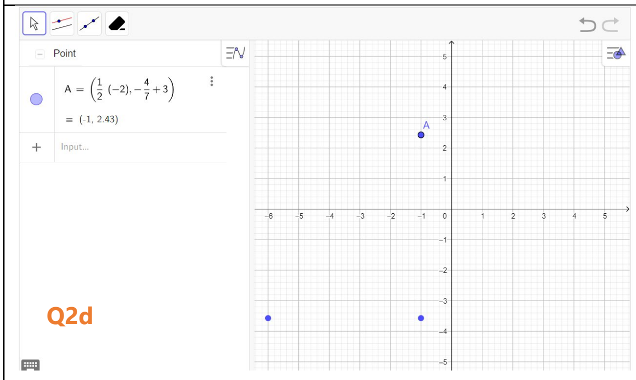
Q2

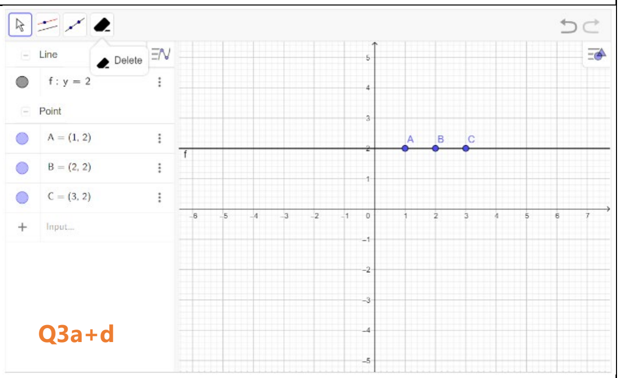
a) Turn on "show trace" by right-clicking on point A.

b) Investigate how you can change the coordinates by dragging the point.

c) Also change point A by entering different values for its x and y coordinates in the algebra view.

d) Change the coordinates by writing mathematical expressions instead of numbers on the *x*- and *y*-coordinates. Challenging yourself to see how complicated a math expression you can come up with?





Q3

a) Create three points that lie on a straight line parallel to the x-axis.

b) Argue why your points lie on a line that is parallel to the x-axis.

Answer guide:

You must argue that something is parallel. Therefore, you must consider what is needed for something to be parallel, write what you find as the first thing in your argument.

• Write something about your points and the line they lie on

• Write something about the x-axis

• Write why this means they are parallel

d) Bonus GeoGebra challenge: Can you construct the line on which the points lie by typing in the algebra window?

Q4

a) Discuss what you can predict about the following points without writing them into GeoGebra.

R= (3,4), U= (3;1,6) and T=(3,-2)

b) Formulate a hypothesis

Answer guide:

A good hypothesis is a claim that something must be true that is not written right before us. The claim that "the x-coordinate of point U is 3" is not a good hypothesis, because it is stated in the assignment. What can you say about the points that you have not already been told?

A hypothesis can start with

We claim that...

It must apply to...

Points R, U and T must...

c) Argue why your hypothesis must be correct.

Answer guide:

What is the mathematical content of your hypothesis that must apply for the hypothesis to be true? Start by writing it.

Then write what you know about the points.

Are there other elements from your hypothesis that you should write about?

Write why this means that the hypothesis is true

Q5

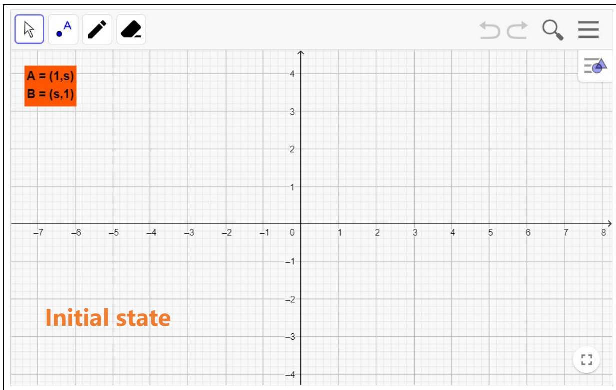
a) Test your hypothesis by entering the coordinate sets for points R, U and T into GeoGebra.

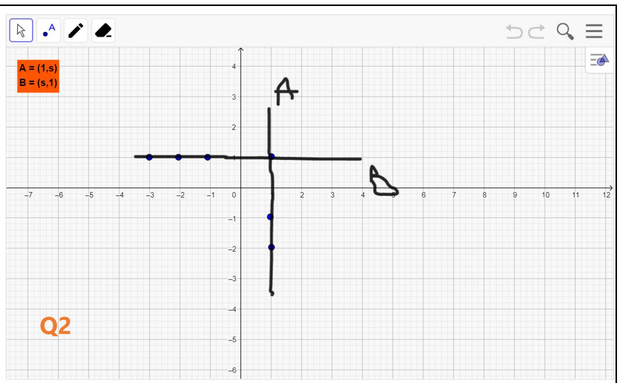
b) Is your hypothesis correct?

If yes, how do you know if your hypothesis is correct?

If no, what did you get wrong in the argument and what did you learn?

Second problem set with one dimensional variable points





Q1

Read:

Points can have a variable in the coordinate set as these two points:

A = (1,s) and B = (s,1) where s is a variable.

Q2

Show and explain how you think points A and B move in the coordinate system when *s* changes value?

(to do so, you can use the tools in the toolbar, and you can also right-click and use the tools there)

Q3

Justify your hypothesis - why do A and B move as you claim?

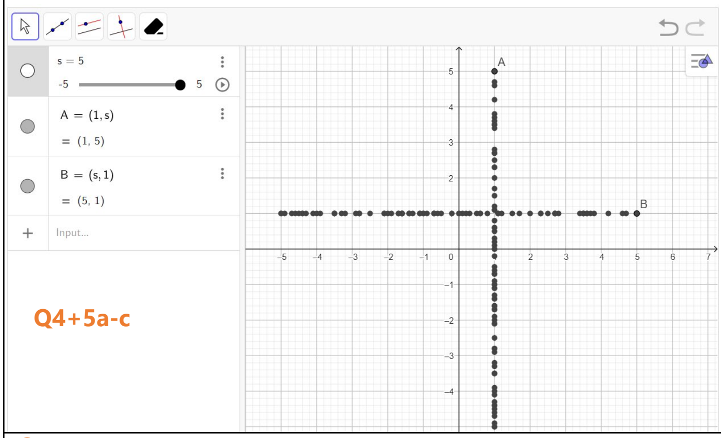
Answer guide:

In question 2 you have shown how you think A and B move.

You must argue why the points move exactly like that.

• Write what you know about the coordinates of the points

• Write why this means that they must move exactly as you say



Q4

Construct the points A = (1,s) and B = (s,1). It is important that you write *s* in the coordinate sets.

Q5

a) Change the value of s by dragging the slider.

b) Turn on "show trace" for the points (right click on the points)

c) Change the value of *s* again by dragging the slider.

d) Explain to the camera how the points move

e) Also explain why they move like that

Q6

a) Formulate a hypothesis about a relationship between the variable *s* and point A

Answer guide:

When a hypothesis is to be about a relationship, it must describe how something affects something else. In this case, it is what happens to A when *s* changes. Feel free to use GeoGebra.

You can, for example, start your hypothesis with:

When s...

If s...

It must apply to A that.... when s....

b) Argue why your hypothesis is true

Answer guide:

What is the mathematical content of your hypothesis that must apply for the hypothesis to be true? Start by writing it.

Then write what you know about the points.

Maybe you should write something about variables?

Or something about the coordinate system?

Write why the things you have mentioned mean that the hypothesis is true.

Q7

c) When does A = B?

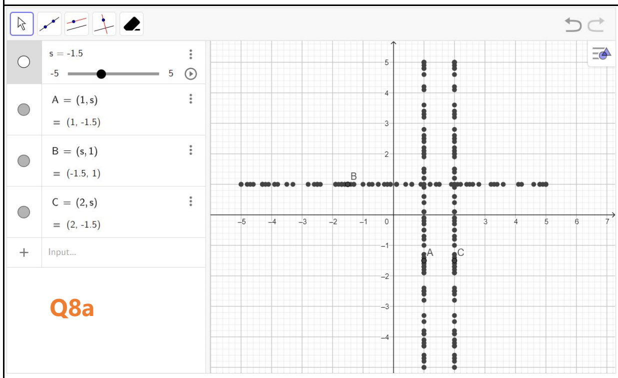
d) What arguments can you come up with to justify when A = B?

Answer guide:

For the answer you must see how many different arguments you can come up with, which justify that your answer in 7a) is correct.

Consider:

- What do you know about the points?
- What do you know about the variable s?
- What can you see?
- Why are the points not the same elsewhere?



Q8

a) Construct a new point C depending on s, which moves parallel to A. (so, *s* must be in the coordinate set of the new point)

b) Can C = A, if so, when? Can C = B, if so, when?

c) Justify your answer

Answer guide:

You must argue why the points are equal or not. Therefore, you must consider what it takes for them to be equal. Write what you find out as the first point in your argument.

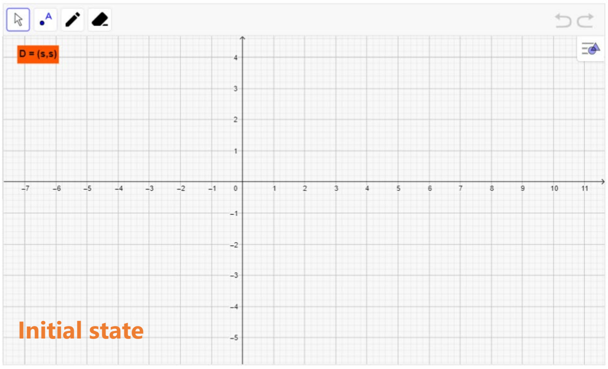
• Write something about point B

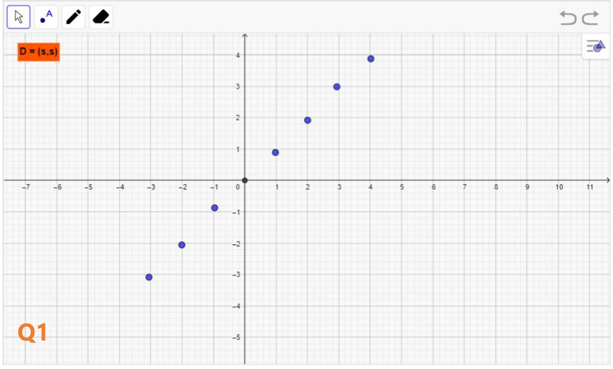
• Write something about Point C

• Write why this means that your answer must be correct

If you find that C cannot be equal to B, try to see if you can change C so that they can. Maybe that can support your argument?

Problem set with two dimensional variable points.





Q1

D = (s,s) and s is a variable.

Show in the coordinate system how point D moves when s changes value (You can use the tools in the toolbar and you can also right-click)

Q2

Justify your hypothesis - why does point D move as you claim?

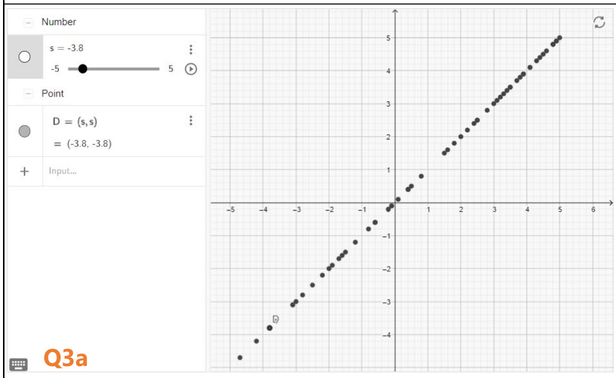
Answer guide:

In question 1 you have shown how you think D moves.

You must argue why D moves exactly like that.

• Write what you know about D

Write why this means that D moves exactly as you say



Q3

a) Type D = (s,s) into the GeoGebra app

b) Change the value of s by dragging the slider

c) Does the point move as you expected?

d) Describe here how the point moves

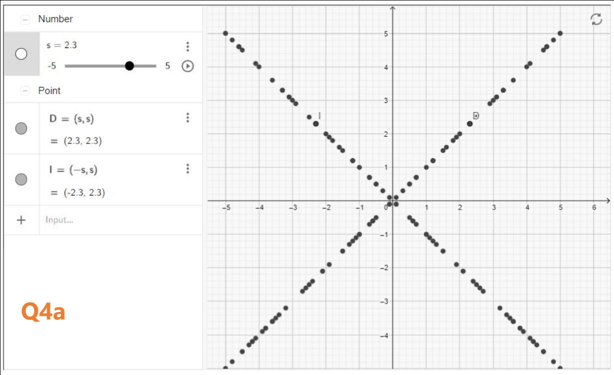
e) Justify why D moves exactly like that

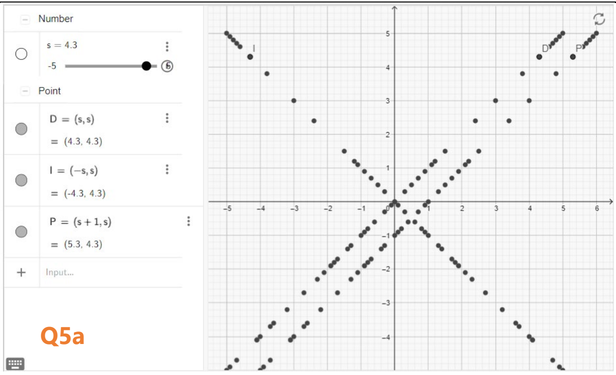
Answer guide:

Consider your answer for question 2. Can you still use the argument after you have seen the point move in GeoGebra?

• If yes, copy it down here. Is there anything that needs to be added or changed?

• If no, formulate a new argument.





Q4

a) Construct a new point I that depends on s and that moves from the 2nd quadrant to the 4th quadrant. (So, you must construct a new point that has s in its coordinate set)

b) Why is your solution correct?

Answer guide:

Your argument must contain the following points:

• What is needed for a point to move from the 2nd quadrant to the 4th quadrant

• What is the coordinate set for I ?

• Why does it mean that you are moving from the 2nd quadrant to the 4th quadrant

Q5

a) Construct a point P dependent on s that never crosses D's path.

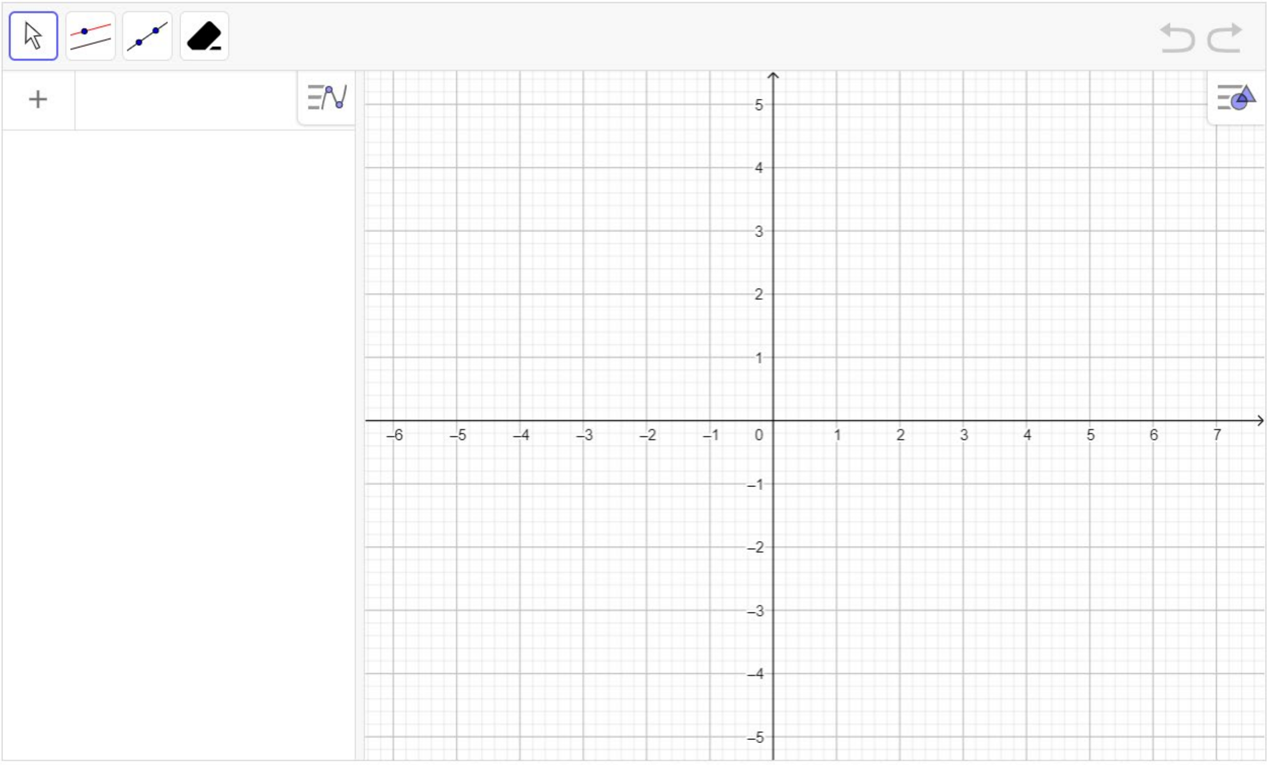
b) What arguments can you find that justify that your point never crosses D's path?

Answer guide:

Your argument must consider the following points:

- What does it take for P to never cross D's path?
- What is the coordinate set for P?
- Why does this mean that P never crosses D's path?

Word document with tasks sequence + online GeoGebra apps Danish



byder at du skal svare i GeoGebra

Opgave 1:

Sp1)

Læs og udfør:

Punkter konstrueres ved at definere deres x- og y-koordinat. Punkter benævnes altid med et stort bogstav.

Prøv selv, indtast følgende: A = (1,2)

Sp2)

a) Slå "vis spor" til, ved at højre klikke på punkt A.

b) Undersøg hvordan I kan ændre koordinaterne ved at trække i punktet.

c) Ændr også på punkt A ved at skrive andre værdier for dens x- og y-koordinaterne i algebravinduet.

d) Ændr koordinaterne ved at skrive regneudtryk (regnestykker) i stedet for tal på x- og y-koordinaterne. Udfordrer jer selv, hvor kompliceret et regneudtryk kan I finde på?

Sp3)

a) Opret tre punkter som ligger på en ret linje parallel med x-aksen.

b) Argumenter for, at jeres punkter ligger på en linje der er parallel med x-aksen.

Jeres svar:	Svar guide: I skal argumentere for at noget er parallelt. Derfor skal I overveje, hvad skal der til for at noget er parallelt, **skriv hvad I finder ud af som det første i jeres argument**. • Skriv noget om jeres punkter og den linje den danner • Skriv noget om x- aksen • Skriv hvorfor det betyder at de er parallelle

c) Argumenter for at jeres punkter ligger på en horisontal linje.

Svar:	Svar guide. I skal argumentere for at noget er horisontalt. Derfor skal I overveje, hvad skal der til for at noget er horisontalt i et koordinatsystem. **skriv hvad I finder ud af som det første I jeres argument**. Skriv noget om jeres punkter og den linje den danner Skriv hvorfor det betyder at den er horisontal

d) Bonus GeoGebra udfordring: Kan I konstruere den linje som punkterne ligger på ved at skrive i algebra vinduet?

Sp4)

a) Diskuter hvad I kan forudsige om følgende punkter **uden** af skrive dem ind i GeoGebra.

R=(3,4) , U= (3;1,6) og T=(3,-2)

b) Skriv en hypotese her under

Jeres svar:	Svar guide: En god hypotese er en påstand om at noget må være sandt, som ikke står skrevet lige foran os. Påstanden at *"x-koordinaten i punkt U er 3"* er ikke en god hypotese, for det står jo i opgaven. Hvad kan I sige om punkterne, som I ikke allerede har fået at vide? En hypotese kan starte med Vi påstår at… Det må gælde at… Punkterne R, U og T må…

c) Argumenter for at jeres hypotese må være rigtig.

Jeres svar:	Svar guide: Hvad er det matematiske i jeres hypotese, som skal gælde for at hypotesen må være sand? **Start med skrive det.** Skriv så hvad I ved om punkterne Er der andre elementer fra jeres hypotese I skal skrive om? Skriv hvorfor det betyder at hypotesen er sand

Sp5)

a) Test jeres hypotese ved at skrive koordinatsættene for punkt R, U og T ind i GeoGebra.

b) Er jeres hypotese rigtig?

Hvis ja, gå til spørgsmål c)

Hvis nej, gå til spørgsmål d)

Opgave 2:

Sp 1) Læs:

Punkter kan have en variabel i koordinatsættet som disse to punkter

A = (1,s) og B = (s,1) hvor s er en variabel.

Sp 2) Vis I koordinatsystemet, hvordan I tror punkt A og B bevæger sig, når s ændrer værdi?

Forklar det også til jeres webcam

(I kan fx bruge værktøjerne i værktøjslinjen og I kan også højre-klikke og bruge de værktøjer der er der)

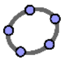
Sp 3) Begrund jeres hypotese - hvorfor bevæger A og B sig som I påstår?

Jeres svar:	Svar guide:
	I spørgsmål 2 har I vist hvordan I tænker A og B bevæger sig. I skal argumentere for hvorfor punkterne bevæger sig lige præcis sådan. • Skriv hvad I ved om punkternes koordinater • Skriv hvorfor det betyder, at de må bevæge sig netop som I siger

Sp 4)

a) Gå til det næste GeoGebra ark i opgave 2 .

b) Konstruer punkterne A = (1,s) og B = (s,1). Det er vigtigt at I skriver s i koordinatsættene.

Sp5) 

a) Ændre på værdien af s, ved at trække i skyderen.

b) Slå "vis spor" til for punkterne (højre klik på punkterne)

c) Ændre igen på værdien af s, ved at trække i skyderen.

d) Forklar til kameraet hvordan punkterne bevæger sig

e) Forklar også hvorfor de bevæger sig sådan

Sp 6)

a) Opstil en hypotese om en sammenhæng mellem variablen s og punkt A

Jeres svar:	Svar guide: Når en hypotese skal handle om en sammenhæng, så skal den beskrive hvordan noget påvirker noget andet. I denne opgave er det hvad der sker med A når s ændre sig. Brug gerne GeoGebra I kan fx starte jeres hypotese med: Når s ... Hvis s ... Det må gælde for A at.... når s....

b) Argumenter for at jeres hypotese er sand

Jeres svar:	Svar guide: Hvad er det matematiske i jeres hypotese, som skal gælde for at hypotesen må være sand? **Start med skrive det.** Skriv så hvad I ved om punkterne Måske I skal skrive noget om variable?

	Eller noget om koordinatsystemet? Skriv hvorfor de ting I har nævnt betyder at hypotesen er sand.

Sp7)

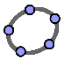
a) Hvornår er A = B?

Jeres svar:	Svar guide: Undersøg hvornår de to punkter er det samme. Skriv kort hvornår A og B er ens.

b) Hvilke argumenter kan I finde for at begrunde hvornår A = B? Noter de argumenter I finder frem til.

Svar:	Svar guide: Her skal I se hvor mange forskellige argumenter I kan finde frem til, som begrunder at jeres svar i 7a) er korrekt. Overvej: Hvad ved I om punkterne Hvad ved I om den variable s Hvad kan I se Hvorfor er punkterne ikke ens andre steder? Brug jeres viden til at forme argumenter.

Sp 8)

a)  Konstruer et nyt punkt C afhængig af s, som bevæger sig parallelt med A. (s skal altså indgå i koordinatsættet i det nye punkt)

b) Kan C = A , hvis ja, hvornår? Kan C = B, hvis ja hvornår?

Jeres Svar:

C = A :

C = B :

c) Argumenter for jeres svar

Jeres svar:	Svar guide:
C = A : C = B :	I skal argumenter for punkternes lighed eller mangel på samme. Derfor skal I overveje, hvad der skal til for at noget er lig med hinanden. **Skriv hvad I finder ud af som det første i jeres argument**. • Skriv noget om punkt B • Skriv noget om Punkt C • Skriv hvorfor det betyder at jeres svar må være korrekt Hvis I finder at C ikke kan være lig med B, forsøg om I kan ændre på C, så de kan. Måske det kan understøtte jeres argument?

Opgave 3:

Find opgave 3 i gruppen.

Sp1)

D = (s,s) og s er en variabel.

Vis I koordinatsystem hvordan punkt D bevæger sig, når s ændre værdi? (I kan bruge værktøjerne i værktøjslinjen og I kan også højreklikke)

Sp2) Begrund jeres hypotese - hvorfor bevæger punkt D sig som I påstår?

Jeres svar:	Svar guide: I spørgsmål 1 har I vist hvordan I tænker D bevæger sig. I skal argumentere for hvorfor D bevæger sig lige præcis sådan. • Skriv hvad I ved om D Skriv hvorfor det betyder, at de må bevæge sig netop som I siger

Sp 3)

a) Skriv D = (s,s) ind i GeoGebra arket (det nederste ark)

b) Ændr på værdien af s, ved at trække i skyderen

c) Opfører punktet sig som I regnede med?

Svar:

d) Beskriv her hvordan punktet bevæger sig:

Svar:

e) Argumenter for hvorfor D bevæger sig netop sådan

Svar:	Svar guide: Se på jeres svar til spørgsmål 2. Kan I forsat bruge argumentet efter I har set punktet I GeoGebra? • Hvis ja så kopier det her ned. Er der noget der skal tilføjes eller ændres? • Hvis nej, så formuler et nyt argument.

Sp4)

a) Konstruer et nyt punkt I afhængig af s, som bevæger sig fra 2. kvadrant til 4. kvadrant. (I skal altså konstruer et nyt punkt som har s i sit koordinatsæt)

b) Hvorfor er jeres løsning korrekt?

Svar:	Svar guide: Jeres argument skal indeholde følgende punkter: • Hvad skal der til for at et punkt bevæger sig fra 2. kvadrant til 4.kvadrant • Hvad er koordinatsættet for I ? • Hvorfor betyder det at I bevæger sig fra 2. kvadrant til 4.kvadrant

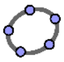
Sp5)

a) Konstruer et punkt P afhængig af s som aldrig krydser D's bane.

b) Hvilke argumenter kan I finde der begrunder at jeres punkt aldrig krydser D's bane?

Svar:	Svar guide: Jeres argument skal indeholde følgende punkter: • Hvad skal der til for at P aldrig krydser D's bane • Hvad er koordinatsættet for P ? • Hvorfor betyder det at P aldrig krydser D's bane

Sp 6)

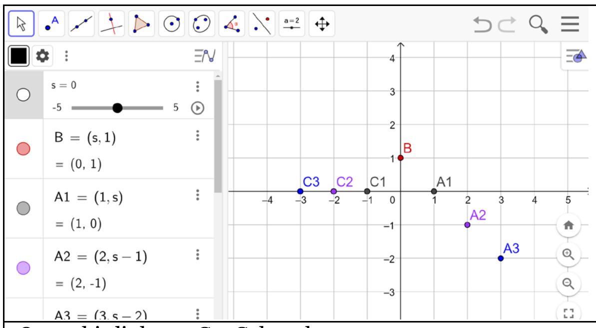
a)  Konstruer et punkt Z afhængig af s som bevæger sig imellem punkt D og P

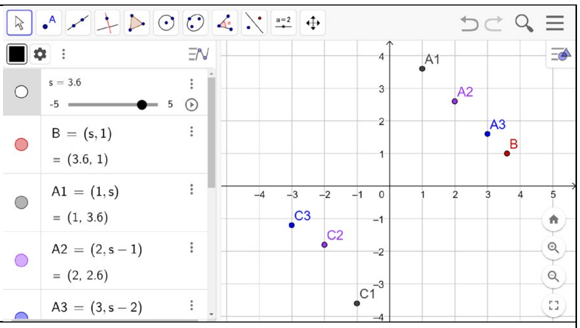
b) Hvorfor er jeres løsning korrekt?

Svar:	Svar guide: I har formuleret mange argumenter nu, så her kan I prøve selv. Er I, i tvivl så se tilbage i opgaverne, måske I kan finde hjælp i jeres tidligere svar.

B: Iteration 3 task sequence

Frist and introductory problem set

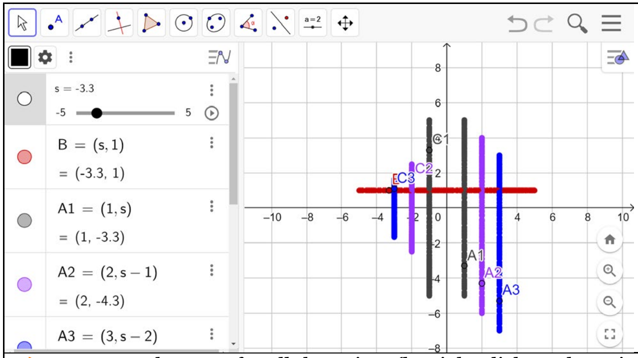




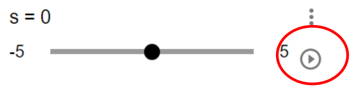
Open this link to a GeoGebra sheet:
https://www.geogebra.org/classic/bps4j3bn

a) Examine by pulling the slider for s:

b) Describe to each other what you see and notice. Try to explain what you see.



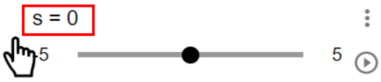
c) Turn on the trace for all the points (by right click on the point). Start the animation on the little arrow.



d) Describe what you are now noticing. Try to explain what you see.

e) Why is it that when you change *s*, the points move? And why do they move differently?

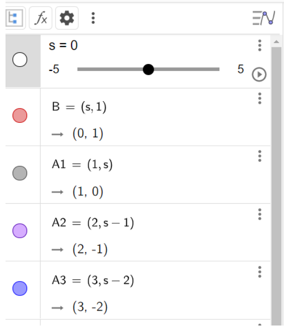
f) One can enter precise values for the variable by clicking s = . (First stop the animation)

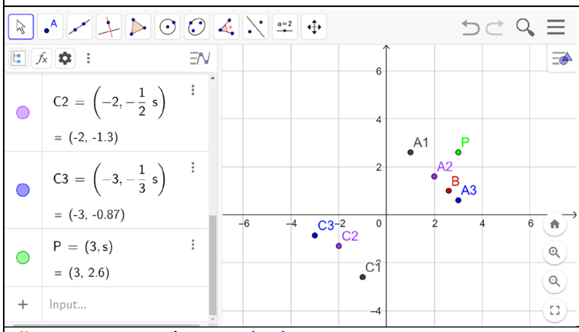


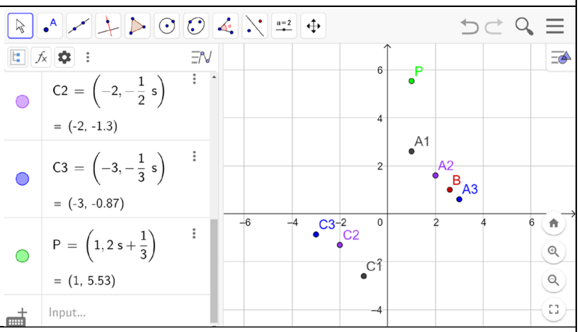
g) Enter different values for *s* and examine the outcome. Describe and explain.

h) Notice the algebra view on the left side. Describe what information you can see there.

i) Each point has two lines of information. Select a point in the algebra view. Describe and explain what the difference is between the two lines? Also examine what happens when you change *s*.







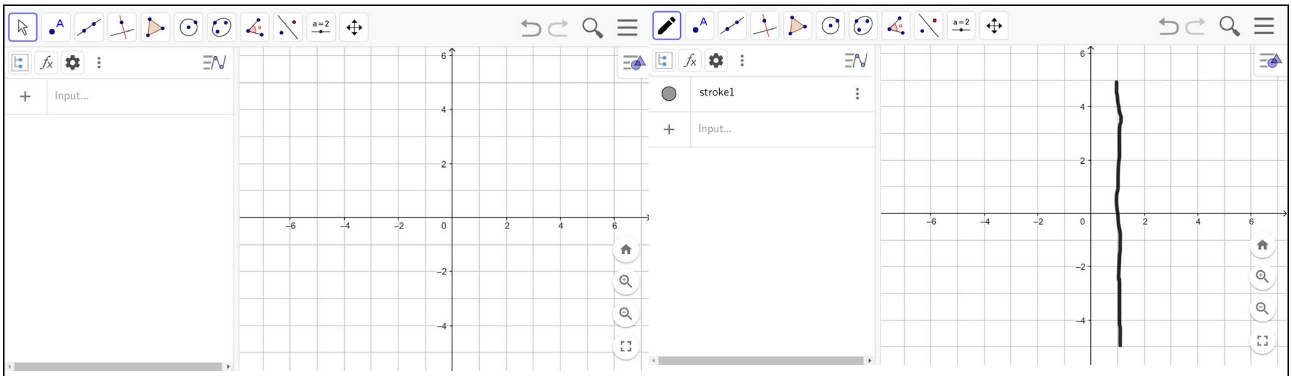
j) Create a point, P = (3,*s*).

- Change P by changing 3 to other values. It can be all kinds of numbers, both natural numbers, decimals, fractions, with exponents - whatever you can come up with. Try several different ones. Remember that in GeoGebra decimals are written with periods and not comma, e.g. 1.2.
- Change P by adding arithmetic operations to *s*. So, +, -, *, /. e.g., P = (3, 0.5s)
- Try combining several operations (e.g., multiplication and minus) e.g., P = (3, 2s-1)

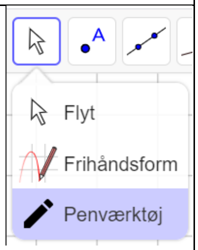
k) Examine, describe, and explain the significance of subtracting/adding to *s*:
Write your answer here:____

l) Examine, describe, and explain the impact of multiplying or dividing *s* by a number.
Write your answer here: ____________________________________

m) What happens when you combine several types of operations?
Write your answer here: ____________________________________

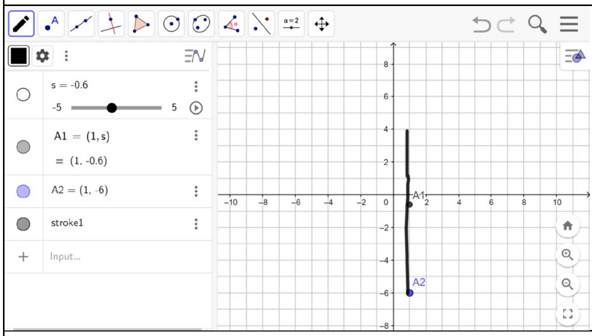


Q1



a) Open a new GGB
b) For this question you can use **the pen tool**, add **points**, lines and the **trace function**.
Guess how A1= (1,s) behaves in the coordinate system when changing the value of *s*.
c) Also show where the track starts and ends.
d) Explain your guess.
e) Now construct the point using the algebra view: A1= (1,s) and examine what happens when you change the value of *s*.
f) Does it match what you thought? Completely, partially, or not at all?" Explain why

When done: If you have drawn, added points or the like, delete all that so that only point A1 remains.



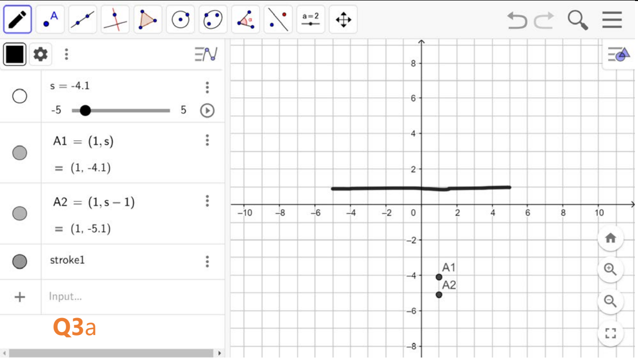
Q2

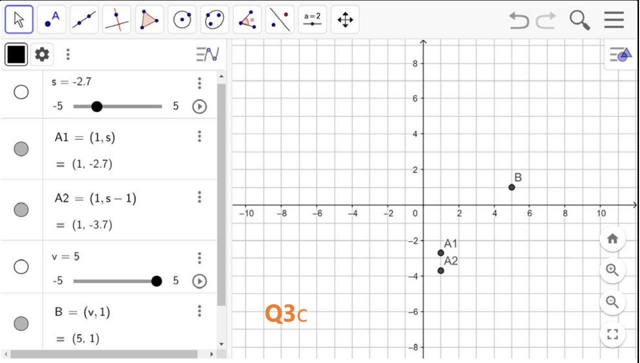
m) For this question you can use **the pen tool**, add **points**, and the **trace function**.
Guess and show how A2 = (1,*s* - 1) behaves in the coordinate system when changing the value of *s*.
n) Also show where the track will start and end.
o) Explain your guess.

p) Explain to each other how A2 is different from point A1.

q) Now construct point: A2 = (1,*s*-1) and examine what happens when you change the value of *s*.
r) Does it match what you thought? Completely, partially or not at all?" Explain why.

When done: If you have drawn, added points or the like, delete all that so that only points A1 and A2 remain.





Q3

a) For this question you can use **the pen tool**, add **points,** and the **trace function**.
Guess how B = (v,1) behaves in the coordinate system when changing the value of v.

i) Explain your guess.
j) Explain to each other how is B different from points A1 and A2? Does B have any similarities with A1 or A2?

k) Now construct point B= (v,1) and examine what happens when you change the value of v.
l) Does it match what you thought? Completely, partially or not at all? Explain why.

m) What is the value of v and s when:

a. B=A1? v=_____ s=________
b. B=A2? v=_____ s=________

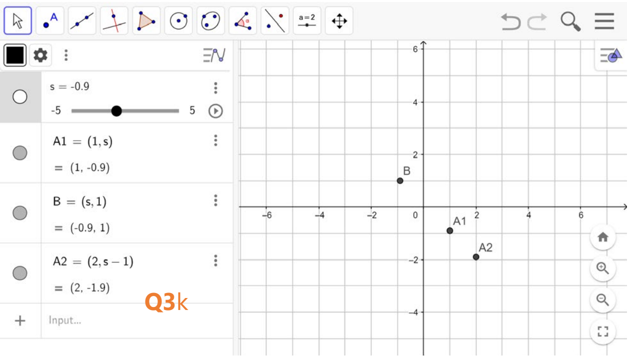
n) Will it also apply if B = (s,1)? Explain why/why not.
o) Change the point B = (v,1) to B = (s,1)
p) Does it match what you thought? Completely, partially or not at all? Explain why.

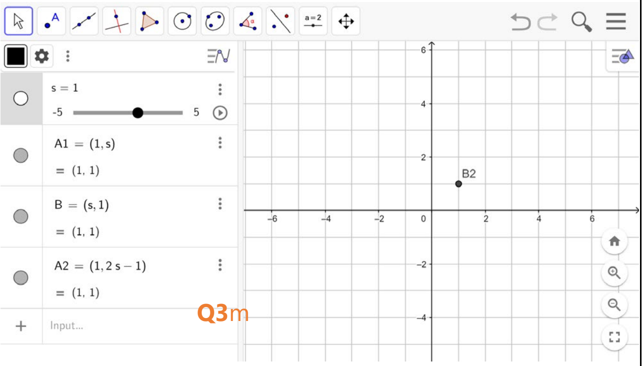
q) Can A1=B? Please mark your answer and explain why/why not.

a. Yes when: ____
b. No.

r) Can A2=B? Please mark your answer and explain why/why not.

a. Yes when: ____
b. No.



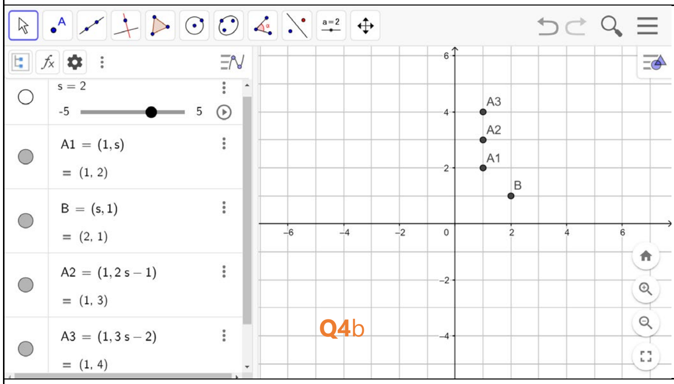


s) If you change A2 ‘s **x-coordinate**. Is it then possible for A2=B?

a. Yes when: ____
Explain why your solution applies.
b. No. Explain why.

t) Change A2 back to A2 = (1,s - 1).
u) Now, if you change A2’s **y- coordinate**, can A2=B without A2=A1?

a. Yes when: ____
Explain why your solution applies.
b. No. Explain why not.



Q4

a) Show and explain: How can you see in the coordinate system that two points:
- are equal to each other **in one point.**
- **always** equal each other
- **are never** equal to each other.

b) Construct a new point A3 dependent on *s* with the following properties (you must construct a new point that has *s* in its coordinate set):
- where *x*=1
- which never equals A1 and A2
- A3 = B in a point

c) Check that your solution meets all three requirements.
d) Explain why your solution applies.
e) Can you formulate a rule for when any point on *x*=1 will be equal B when *s* = 1?

Answer:____

Q5

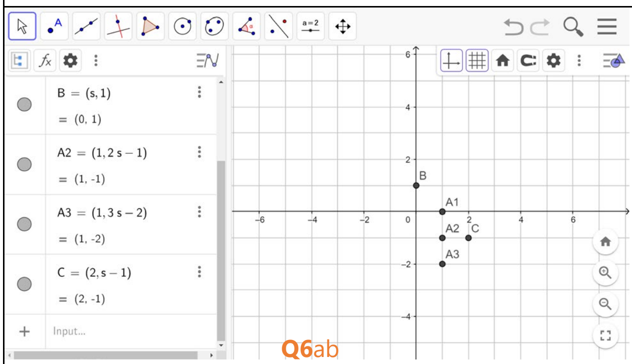
This task is a thought experiment.

a) In theory, how long can the tracks of a point be? Is there any limitation? Explain why/why not.
b) Now if we imagine that there is such a track on the line x=1. Which of these points are on that track? (Mark which points)

(1,s) (3,s) (1, 4s-100) (2,4s+100) (1, 1/2s) (s3,1) (s/s, 4) (s/s, s·s) (s/s, s^s)

c) Formulate a rule for which points are on the track? Explain why your rule applies.

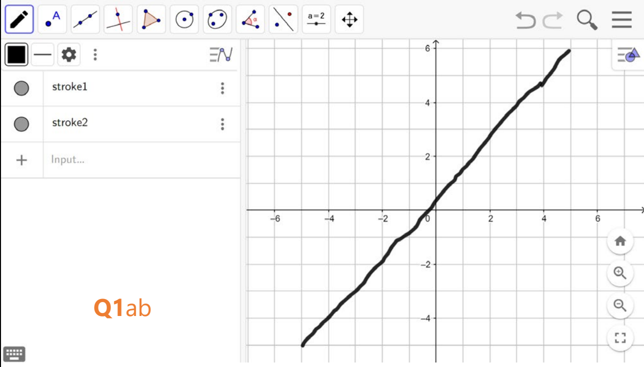
Answer:____

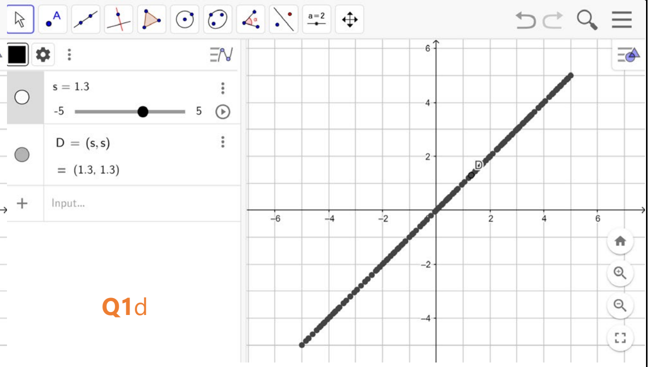


Q6

a) Construct a new point C depending on *s* that moves parallel to A1 and A2.
b) Can C=B? If so, when? If not, is it possible to change C so C =B?
c) Explain why

Second problem set with two dimensional variable points





Q1

Open a new GGB

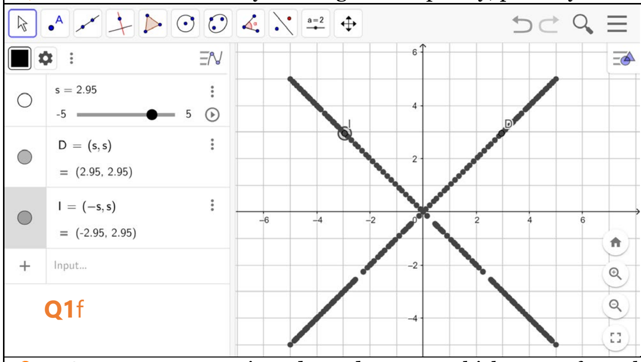
a) For this question you can use **the pen tool**, add **points,** and the **trace function**.
Guess how D = (s,s) behaves in the coordinate system when changing the value of *s*.

b) Also show where the track starts and ends.

c) Explain your guess.

d) Now construct the point: D= (s,s) and examine what happens when you change the value of *s*.

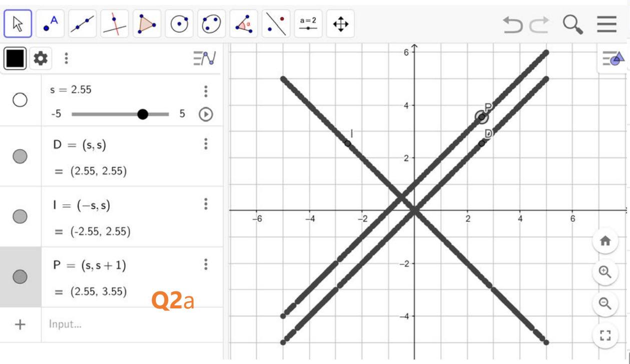
e) Does it match what you thought? Completely, partially or not at all? Explain why

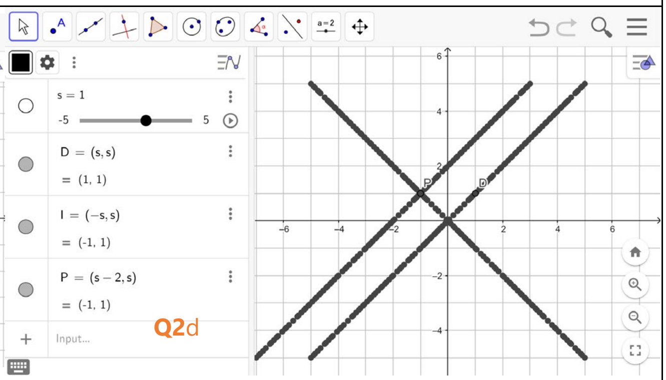


f) Construct a new point I dependent on s, which moves from the 2nd quadrant to the 4th quadrant. Why is your solution correct?

g) Can D = I, if so, when? Please mark your answer and explain it.

- Yes when:____
- No.





Q2 a) Construct a point P depending on s which never intersects D's trajectory. b) Explain why your point P does not intersect D. c) Can P = I? If so, when? Please tick your answers. Explain why/why not to each other. a. Yes when:____ b. No d) If you now change the coordinates of P, is it possible that P = I? If so, when e) Explain why your solution
Q3 Open the same file from earlier in GeoGebra. https://www.geogebra.org/classic/bps4j3bn Examine it again, is there anything new you notice or something you can better explain now?

C: Abbreviations

RC – Reasoning Competency

IAME – Instrumental approach to mathematics education

IA – The instrumental approach, also known as the theory of instrumental genesis

IJ – Instrumented justification

DR – Design based research

DGE – Dynamic Geometry environments

DGAE – Dynamic geometry and algebra environments

CAS – Computer algebra system

KOM – The KOM-framework: Competencies and Mathematical Learning

NT – Networking of theories

ATD - Anthropological theory of didactics

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